THE FIFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

WITH the Annual Meeting held on Wednesday, December 28, 1898, the AMERICAN MATHEMATICAL SOCIETY completed the tenth year of its continuous existence under its present and its former name, having been founded as the New York Mathematical Society on November 24, 1888. Its reorganization as a national society went into effect of date July 1, 1894. Reminiscences of the earlier days and a forecast of the aims and prospects of the widened organization were the theme of Dr. Emory McClintock's presidential address: "The past and future of the Society," delivered at the annual meeting, December 28, 1894, and published in the BULLETIN for January, 1895. The rapid growth of the Society in numbers and influence, in scientific zeal and productivity, is steadily realizing the high aspirations of its supporters and friends. It is safe to say that no scientific body in the country is in a more healthy and vigorous condition. The membership has increased from 135 in May, 1891, to 244 in September, 1894, and to 316 in January, 1899. The number of papers presented in 1898 was 88, against 24 in 1894, and the standard of excellence has been at least maintained. The total attendance of members at the seven meetings of 1898 was 182.

An especially encouraging feature is the large number of members who succeed in attending at least one meeting during the year. Nearly one-third of the members, 104, paid this tribute in 1898 to their interest in the work of the Society. This is certainly a large proportion, considering that the meetings were held only in New York, Chicago and Boston, while the members are busy teachers and professional men scattered over the entire country, the foreign membership also being considerable.

Two factors have contributed powerfully to increase the Society's strength and growth since its reorganization as a national body. One of these is the institution of summer meetings, held usually in connection with the large general scientific gatherings, and thus offering special convenience of time and place. The summer meetings have proved extremely popular and interesting, and together with the colloquia by which they have twice been reinforced, they have done more than any other one influence to unite the workers and teachers of mathematics throughout the country in a

common bond of professional interest and sympathy. The second strengthening factor was the founding of the Chicago Section, which loyally represents the interests and disseminates the influence of the Society in the West. The aims of the Section are in perfect harmony with those of the Society. Its meetings, which have thus far been held twice a year, bring together and afford a welcome stimulus to many members to whom the New York meetings would usually be inaccessible. The Section has been a focus of great scientific activity, and its work has reflected great credit on the Society.

The attendance at the Annual Meeting numbered twenty-eight and included the following members of the Society:

Professor E. W. Brown, Dr. A. S. Chessin, Professor F. N. Cole, Professor T. S. Fiske, Mr. G. B. Germann, Professor Harold Jacoby, Professor W. W. Landis, Professor Gustave Legras, Dr. G. A. Miller, Professor Simon Newcomb, Professor G. D. Olds, Mr. J. C. Pfister, Professor M. I. Pupin, Miss Amy Rayson, Professor J. K. Rees, Dr. Virgil Snyder, Professor J. H. Tanner, Miss Mary Underhill, Professor J. H. Van Amringe, Professor J. M. Van Vleck, Professor L. A. Wait, Professor R. S. Woodward.

The President of the Society, Professor Simon Newcomb, occupied the chair during the two sessions, which began at 10.30 a. m. and 2.30 p. m. The Council announced the election of the following persons to membership in the Society: Dr. Charles L. Bouton, Harvard University, Cambridge, Mass.; Dr. Adolph A. Himowich, New York, N. Y.; Dr. Francis S. Macaulay, St. Paul's School, London, England; Professor Buzz M. Walker, State Agricultural and Mechanical College, Miss. Four applications for membership were received. Reports were presented by the Treasurer and the Librarian. These reports will be printed in the List of Members now in preparation. A committee consisting of Professor F. N. Cole, T. S. Fiske, H. S. White and R. S. Woodward, were appointed by the Council to arrange for the summer meeting of the Society.

At the annual election the following officers were chosen

President,	Professor R. S. WOODWARD,
First Vice-President,	Professor E. H. MOORE,
Second Vice-President,	Professor T. S. FISKE,
Secretary,	Professor F. N. COLE,
Treasurer,	Professor HAROLD JACOBY,
Librarian,	Professor POMEROY LADUE.

Committee of Publication,

Professor T. S. FISKE,
Professor F. N. COLE,
Professor ALEXANDER ZIWET.

Members of the Council to serve until December, 1901,

Professor MAXIME BÔCHER,
Professor JAMES PIERPONT,
Professor CHARLOTTE ANGAS SCOTT.

The following papers were presented:

(1) Professor M. I. PUPIN: "On multiple resonance."

(2) Dr. A. S. CHESSIN: "On the development of the perturbative function in terms of the eccentric anomalies."

(3) Dr. A. S. CHESSIN: "On some points of the theory of functions."

(4) Professor E. O. LOVETT: "On the transformation of straight lines into spheres."

(5) Dr. E. J. WILCZYNSKI: "A generalization of Appell's factorial functions."

(6) Dr. VIRGIL SNYDER: "Asymptotic lines on ruled surfaces having two rectilinear generators."

(7) Dr. G. A. MILLER: "On a memoir on the substitution groups whose degree is less than nine."

(8) Dr. W. SCHULZ: "On the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ke^u$$

and its connection with Dirichlet's principle."

(9) Professor ORMOND STONE: "On the solution of Delaunay's canonical system of equations."

In the absence of the authors, Professor Lovett's paper was read by title, Dr. Wilczynski's was presented by Professor Fiske, and Professor Stone's by Professor Woodward. Dr. Wilczynski's paper was offered to the Society through Dr. L. E. Dickson, Dr. Schulz's through Professor J. H. Tanner. Abstracts of those papers not intended for publication in the BULLETIN are given below.

Professor Pupin discussed the forced vibrations of a multiplex under the action of a harmonically varying force.

In a series of papers published in the *Astronomical Journal* (1894, Nos. 326 and 332; 1898, Nos. 442 and 452) Dr.

Chessin has given a method by which the computation of the terms in the development of the perturbative function is greatly simplified. The object of the present paper is to indicate a further simplification for the case of the development in terms of the eccentric anomalies as given by Professor Simon Newcomb in his Astronomical Papers, vol. III, part 1. This simplification consists in applying the symbols $\prod_{n,n'}^{m,m'}$ not to the coefficients $a'A_i$; $a'B_i$; $a'C_i$; ··· of Professor Newcomb but to these coefficients multiplied by $(1+\varepsilon^2)^{-\mu}(1+\varepsilon'^2)^{-\nu}$, the values of μ and ν being respectively equal to $-i$, i; $-i+1$, $i+1$; $-i+2$, $i+2$; ···. If then we denote by $X_{m,n}^{m,n}$, $Y_{m,n}^{m,n}$, $Z_{m,n}^{m,n}$, ··· the products of the coefficients $P_{m,n}^{m,n}$, $Q_{m,n}^{m,n}$, $R_{m,n}^{m,n}$, ··· by $(1+\varepsilon^2)^{-m}(1+\varepsilon'^2)^{-n}$ respectively, we obtain a development of the perturbative function which differs from Professor Newcomb's only in the substitution of the X, Y, Z, ··· for the P, Q, R, ··· and which presents this advantage over it: while the computation of the X, Y, Z, ··· with equal upper and lower indices is not perceptibly different from the computation of the corresponding P, Q, R, ···; the computation of the terms with different upper and lower indices can be made by considerably simpler formulas than in the case when the P, Q, R, ··· are used.

From the time that Galois proved that the theory of equations is based upon the theory of substitution groups the determination of all the possible substitution groups of a given degree has been a problem of fundamental importance in algebra. During recent years a large number of substitution groups, usually found by tentative processes, have been published. The first part of Dr. Miller's contemplated memoir is devoted to the theory of the construction of substitution groups. This theory is developed to a sufficient extent to include all the groups whose degree does not exceed nine. The necessary and sufficient condition that a given abstract group may be represented as an intransitive substitution group, not simply isomorphic to one of its transitive constituents, is that it contains at least two self-conjugate subgroups, differing from identity, which have only identity in common. The construction of all the imprimitive groups that contain substitutions besides identity which leave all the systems of imprimitivity unchanged can be reduced to the construction of transitive groups containing given intransitive groups as self-conjugate subgroups and permitting the systems of intransitivity of

this self-conjugate subgroup according to known substitution groups. If an imprimitive group is simply isomorphic to the group according to which its systems of imprimitivity are permitted, its abstract group properties are known and hence its forms as a substitution group are evident. The second part of the memoir is devoted to the determination of all the possible substitution groups whose degree is less than nine by means of the theorems developed in the first part. The notation employed exhibits what groups are simply isomorphic and hence may be helpful to the student of abstract groups. Brief historical notes on some of the best known groups are given.

Dr. Schulz introduces the two complex variables

$$z = x + yi, \quad z_0 = x - yi$$

into the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ke^u$$

and defines

$$u = u_0 + zz_0u_1 + \cdots + (zz_0)^n u_n + \cdots,$$

where $u_0, u_1, \cdots, u_n, \cdots$ satisfy the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and are each the sum of two conjugate functions of z and z_0. The principal results are the following: The functions $u_1, u_2, \cdots u_n, \cdots$ (n finite) are absolutely convergent in the same region as u_0, and are uniformly defined by u_0. They tend toward continuous one valued boundary functions with the period 2π, if u_0 is defined by such a boundary function according to Dirichlet's principle. A theorem here introduced represents a distinct advance in the theory of the potential function. The function v_0 conjugate to u_0 also tends toward a continuous limit on the boundary whether u_0 does or does not have a derivative on the boundary. The necessary condition for the convergence of the series for u is that u_0 is defined for a circle of radius equal to or less than a fixed finite quantity. This quantity can be expressed by the boundary values of u_0. If the largest absolute value of the boundary function of u_0 is less than a fixed finite quantity, the function u will be absolutely convergent in the same region of convergence as u_0, and the function u

will be absolutely convergent for a smaller region than u_0. The function u tends to a continuous limit on the boundary of a circle in both cases of convergence and the function u with its boundary function is uniformly defined by the boundary function of u_0. By conformal representation the theorem of convergence can be extended to a simply connected surface with any boundary whatever. Assuming that the boundary is composed of analytic curves with only a finite number of singular points, it is proved that u tends to a continuous limit on the boundary except at singular points where logarithmic discontinuities occur. Finally the boundary functions of u and u_0 correspond to each other uniformly.

In the *Proceedings of the London Mathematical Society* (Vol. 27, p. 385), Professor E. W. Brown gives a simple and interesting solution of Delaunay's canonical system of equations by means of Hamilton's principal function. In Professor Stone's solution Hamilton's function is avoided.

F. N. COLE.

COLUMBIA UNIVERSITY.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Thursday and Friday, December 29 and 30, 1898. The attendance and number of papers contributed indicate the interest, in the work of the Society and in the development of mathematical science, of those members who are accustomed to attend the meetings of this Section, and would seem to clearly justify its formation.

The total attendance numbered twenty-six including the following members of the Society:

Professor Henry Benner, Professor E. W. Davis, Dr. Harris Hancock, Professor T. F. Holgate, Mr. H. G. Keppel, Professor Malcolm McNeill, Professor H. Maschke, Professor E. H. Moore, Professor H. B. Newson, Professor James Pierpont, Professor J. B. Shaw, Dr. H. F. Stecker, Professor A. L. P. Wernicke, Professor H. S. White, Mr. Wm. H. Williams, Professor Mary F. Winston, Professor J. W. A. Young.

The first session opened at 10 o'clock on Thursday with Professor E. H. Moore, Vice-President of the Society, in the chair. Aside from the reading and discussion of papers the Section gave attention to a few items of necessary business. Professor T. F. Holgate was re-elected Secretary for the ensuing year and Professors J. B. Shaw and E. W. Davis were chosen as additional members of the Programme Committee. The next meeting of the Section will be held on Saturday, April 1, 1899, at Northwestern University, Evanston.

The following papers were read:

(1) Professor J. W. A. YOUNG: "Report on the teaching of mathematics in the higher schools of Prussia."

(2) Professor H. B. NEWSON: "On the Riemann-Helmholtz problem."

(3) Professor H. S. WHITE: "Preliminary report on certain new relations among the fundamental covariants of a ternary cubic."

(4) Professor E. H. MOORE: "The decomposition of modular systems connected with the doubly generalized Fermat theorem. (First communication.)"

(5) Professor E. H. MOORE: "Concerning Klein's group of $(n+1)!$ n-ary collineations."

(6) Professor J. B. SHAW: "Some quaternion integrals and their related classes of functions."

(7) Professor H. B. NEWSON: "Normal forms of projective transformations (second communication)."

(8) Professor H. MASCHKE: "Some general theorems concerning linear substitution groups of finite order."

(9) Dr. H. F. STECKER: "Non-Euclidean images of plane cubics on rotation surfaces of constant negative curvature."

(10) Professor H. B. NEWSON: "What constitutes a one parameter group?"

(11) Dr. L. E. DICKSON: "The determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant, with the announcement of two new systems of simple groups."

(12) Mr. C. C. ENGBERG: "The Cartesian oval and the auxiliary parabola."

(13) Professor T. P. HALL: "An algebra of space and its relation to quaternions (preliminary communication)."

In the absence of Dr. Dickson, Mr. Engberg and Professor Hall their papers were presented and read by title.

In Professor Newson's paper (No. 2) the attempt was made to find the properties common to the system of

Euclidean and the two systems of non-Euclidean motions and those by which these systems of motions are distinguished from all other possible systems. The motion of a rigid body in threefold space is a point to point transformation; it is furthermore a projective transformation, since lines are transformed into lines and planes into planes; it is also a conformal transformation, since all angles are transformed into equal angles. The transformations which are common to G_{15}, the general projective group, and G_{10}, the conformal group, in threefold space are either motions or transformations of similarity. These groups have in common three subgroups, viz.: G_7 the group of similarity, $G_6(H)$ and $G_6(E)$, where H is the absolute of hyperbolic space and E that of elliptic space. The invariant figure of G_7 is the absolute of parabolic space. The common and characteristic properties of these three groups can be found without difficulty.

The following is a brief outline of the preliminary report (No. 3) by Professor White: Gordan's reduced form system of the ternary cubic contains 34 forms, which can be arranged by order and class into 16 groups; two are invariants, four covariants (in the restricted sense), four contravariants, one the identical covariant, and twenty-three mixed concomitants. It is proposed as desirable to tabulate the reduced form system of each of these considered as a new stem form, then to set up the relatively small simultaneous systems of these 34 forms taken in pairs, etc. As a contribution to this end, in addition to results published by Clebsch and Gordan in Volume 6 of the *Mathematische Annalen*, certain covariants are calculated and reduced which arise from the seven forms which are of equal order and class. The chief interest in these particular forms lies in their aspect as covariant operators with recurrent laws of low degree.

Paper No. 4 stated a converse of the theorem given by Professor Moore in the BULLETIN for April, 1896.

Paper No. 5, also by Professor Moore, effects the determination, for various geometrical interpretations, of the fundamental region in real space of the group of substitutions

$$y_i = y'_{a_i} \qquad (i = 0, \cdots, n)$$

where the indices $a_0, \cdots, a_n$ are in some order $0, \cdots, n$.

In Professor Newson's second communication on projective transformations (No. 7) all the different types of projective transformations in one, two, and threefold space are reduced to their normal forms in terms of their essential parameters, *i. e.*, the coördinates of their invariant elements and their characteristic cross ratios.

Professor Maschke gave in his paper (No. 8): First, a short report of the theorem: "If in a finite linear substitution group G at least one coefficient occupying one and the same place (but not in the principal diagonal) in the matrices of the substitutions of G, vanishes in every substitution, then G is intransitive." This part of the paper has been offered to the *Mathematische Annalen* for publication. Second, a report of the theorem: "If at least one substitution of a finite linear substitution group possesses one root which is different from its other roots, then the group can be so transformed that every coefficient of every substitution of G is cyclotomic." This theorem is an extension of a similar theorem which was given in a paper read before the Society at the Toronto meeting 1897 (*Mathematische Annalen*, vol. 50, p. 452). Third, a geometrical proof of the intransitivity of every finite quaternary substitution group, each substitution of which possesses two double roots.

In Dr. Stecker's paper (No. 9) certain geometrical material was presented which the author hopes to make use of in his further endeavors to derive geometrically the non-Euclidean properties of a plane cubic. This material related to the hyperbolic case, similar material already existing for the elliptic case. By a non-Euclidean image of a plane cubic is understood a curve or part of a curve whose parabolic properties, properly interpreted, apply to the elliptic or hyperbolic plane. The equations and forms of portions of certain sextic curves (corresponding to the five types of ordinary plane cubics) were derived by stereographic projection of the parabolic plane upon the sphere, followed by an orthographic projection of the lower half sphere. These parts of sextic curves were then built upon the three types of rotation surfaces of constant negative curvature, giving the material sought.

Dr. Dickson's paper (No. 11) is intended for publication in the *American Journal of Mathematics*. The following abstract will serve to characterize it:

Every quadric form in m variables with coefficients belonging to the Galois field of order $p^n (p > 2)$ and having a de-

terminant not zero in the field, can be reduced by a linear homogeneous m-ary substitution in the field, to one of the two canonical forms

$$\sum_{i=1}^{m} \xi_i^2, \quad \sum_{i=1}^{m-1} \xi_i^2 + \nu \xi_m^2,$$

where ν is a not-square in the field. The second form is reducible to the first if m be even. The first form defines the orthogonal group, the structure of which was announced by the writer in the BULLETIN for May, 1898. For the case $m = 4$ the quotient groups of composite order are isomorphic to the simple group of linear fractional substitutions of determinant unity in the Galois field of order p^n. The second form defines a new group leading to a system of simple groups of order

$$\tfrac{1}{2}[p^{nk} + (\pm 1)^k] p^{n(k-1)} (p^{n(2k-2)} - 1) p^{n(2k-3)} \cdots (p^{2n} - 1) p^n,$$

where $m = 2k > 2$ and the sign $\pm$ depends upon the form $4l \pm 1$ of p^n. For $m = 2k = 4$, this simple group of order $\frac{1}{2}(p^{4n} - 1)p^{2n}$ and belonging to the $GF(p^n)$ is isomorphic to the simple group of linear fractional substitutions of determinant unity in the $GF(p^{2n})$.

Every m-ary quadratic form in the Galois field of order 2^n, not expressible in terms of fewer than m variables in the field, can be reduced by a linear homogeneous m-ary substitution in the field, to one of the three canonical forms

$$\xi_0^2 + \sum_{i=1}^{M} \xi_i \eta_i, \quad \sum_{i=1}^{M} \xi_i \eta_i, \quad \sum_{i=1}^{M} \xi_i \eta_i + \lambda \xi_1^2 + \lambda \eta_1^2,$$

there being one type if m is odd and two types if m is even. The group defined by the first form is simply isomorphic to the Abelian group of $2M$ indices in the $GF(2^n)$ and is *simple* if $M > 1$, the case $M = 2$, $n = 1$ being an exception. The group defined by the second and third forms are respectively the generalized first and second hypoabelian groups. In the third form λ is any mark such that $\xi_1\eta_1 + \lambda\xi_1^2 + \lambda\eta_1^2$ is irreducible in the field. The generalization of Jordan's second hypoabelian group is here first announced. The simple groups obtained by its decomposition are of order

$$(2^{mn} + 1)[(2^{2n(m-1)} - 1)2^{2n(m-1)}] \cdots [(2^{2n} - 1)2^{2n}],$$

m being any even integer and n any integer whatever.

Mr. Engberg's paper (No. 12) was a minute and detailed

discussion of the Cartesian oval and its accompanying parabola.

Professor Hall's paper (No. 13) was a continuation of the subject treated in his paper read before the Society at the Boston summer meeting.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILLINOIS.

REPORT ON RECENT PROGRESS IN THE THEORY OF THE GROUPS OF A FINITE ORDER.*

BY DR. G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Boston, August 25, 1898.)

DURING the last decade the general theory of groups has been made much more accessible by means of the publication of a number of treatises. Among these the six volumes by Lie, assisted by Engel and Scheffers (1888–1896), stand out preëminently. The other important treatises that were published in this period are: Cole's translation of a revised edition of Netto's "Theory of Substitutions" (1892); Kantor, "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene" (1895);† Vogt, "Leçons sur la résolution algébrique des équations" (1895); Weber, "Lehrbuch der Algebra" (1895–1897); Burnside, "Theory of Groups of a Finite Order" (1897); Klein-Fricke, "Automorphe Functionen (Die gruppen-theoretischen Grundlagen)" (1897); Bianchi, "Teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Galois" (1897).

A number of other books published during this period

* The paper was prepared on the invitation of the officers and committee of Section A, "with a view to obtaining at this anniversary meeting such a survey of the field as may lead to a possible coöperation of effort."

† Cf. Wiman, *Math. Annalen*, vol. 48 (1896), pp. 195–240. This article has for its object the complete enumeration of the finite groups of birational transformations in the plane. The results do not agree with those at which Kantor arrived in the treatise cited.

develop this theory less extensively with a view to some direct applications. Among these are: Borel-Drach, "Théorie des nombres et de l'algèbre supérieure" (1895); Picard, "Traité d'analyse, vol. 3" (1896); Page, "Ordinary differential equations; An elementary text-book, with an introduction to Lie's theory of the group of one parameter" (1897). In Picard's Traité the theory of substitutions is introduced with a view towards studying the analogies between the theory of linear differential equations and the Galois theory of algebraic equations. This is a departure from the methods followed in the older works on this subject.

In view of the large number of these recent treatises and the extensive reviews which have been devoted to them in journals that are easily accessible, it seems permissible to confine this report to a few subjects which seem to offer inviting fields for investigation. It is the object to avoid, as far as practicable, the consideration of the recent developments which are fully treated in any of the given works and to aim to supplement a few of them rather than to state what they contain.

We shall generally confine ourselves to the consideration of developments which have been published during the last five years, stating only so much of the earlier results as may seem necessary to give a clear view of the problems. The vastness of our subject may be inferred from the following statement of Lie: *The concepts, group and invariant, take each day a more preponderant place in mathematics and tend to dominate this entire science.** Unexpected demands on the writer's time have compelled him to confine his attention to the subjects which relate to groups of a finite order.

§1. *Solvable groups.*†

A solvable group of composite order must be compound, but a compound group is not necessarily solvable. Sometimes the proof that all the groups of composite orders that belong to a given system are compound is equivalent to the proof that all these groups are solvable. This is clearly the case when the self-conjugate subgroups and the corresponding quotient groups belong to the same system, *e. g.*, when Frobenius proved that every group of a composite order that is not divisible by the square of a prime number must be

* "Influence de Galois sur le développement des mathématiques," *Le Centenaire de l'École Normale*, 1895, p. 485.

† In Weber's recent Algebra these groups are called "metacyclic," vol. 1, p. 598.

compound he also proved that all these groups are solvable, for the orders of their self-conjugate subgroups and of their quotient groups cannot be divisible by the square of a prime number. Similarly, the proof that every group whose order is a power of a prime number is compound proves at the same time the solvability of all these groups.

On the contrary, a group whose order is divisible by 2 without being divisible by 4 must contain negative substitutions when it is represented as a regular group and hence it must be compound when its order exceeds 2. This, however, does not prove that such a group must be solvable, unless we could assume that every group of an odd order is solvable. Burnside has recently called attention to the fact that it would be very desirable to prove either the existence or the non-existence of a simple group of an odd order. If such a group exists it cannot be represented as a doubly transitive substitution group.

From the preceding paragraph we see that the proof that every group of an odd order is compound (if this proof is possible) would prove, at the same time, the solvability of every group whose order is not divisible by 4. It has been observed that the groups of order $p^\alpha q^\beta$ (p and q being prime numbers) furnish an almost equally interesting field for study. Frobenius proved* that such groups are solvable whenever $\beta = 1$ and Jordan published an article† a few months ago in which he proved that they are also solvable whenever $\beta = 2$. When $\beta = 0$ we have groups whose order is a power of a prime number. Sylow proved‡ a long time ago that all these groups are solvable. Many special cases of groups whose order is of the form $p^\alpha q^\beta$ have been studied.§

Within the last two years two special systems of solvable groups have been investigated. One of these systems was first considered by Dedekind and has been called by him the Hamiltonian groups‖. It includes all those groups which are non-Abelian and contain only self-conjugate subgroups. The group of lowest order that belongs to this system is the quaternion group, *i. e.*, the group of order 8 which contains 6 operators of order 4. Dedekind found that all the groups of this system have a surprisingly simple structure, viz., each of them is the direct product of the qua-

* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 185.
† Jordan, *Liouville's Journal*, vol. 4 (1898), p. 21.
‡ Sylow, *Math. Annalen*, vol. 5 (1872), p. 588.
§ Cf. Burnside, Theory of Groups, 1897, Ch. XV.
‖ Dedekind, *Math. Annalen*, vol. 48 (1897), p. 548.

ternion group and an Abelian group which includes only the operators that are commutative to every operator of the group. The writer has studied this infinite system somewhat farther and arrived at a number of new results.* One of the most important of these is that there is one and only one Hamiltonian group of order 2^a, whenever $a > 2$.

A somewhat more extensive system which includes the preceding was studied by Ahrens with a view to pointing out analogies between Lie's theories and the theories of the groups of a finite order.† This system includes all the groups that possess the property that we arrive at identity by forming the successive groups of cogredient isomorphisms. That all the groups of this infinite system are solvable follows directly from the fact that the factors of composition of any group may be found by combining the factors of composition of its group of cogredient isomorphisms and the prime factors of the quotient obtained by dividing its order by that of this group of cogredient isomorphisms.

Although this system includes an infinite number of solvable groups yet there are evidently many infinite systems of solvable groups which do not belong to it. As such systems we may mention the almost trivial systems obtained by forming the direct products of the symmetric groups of degrees three or four and all the possible Abelian groups. From the fact that every group whose order is a power of a prime number includes operators that are commutative to every operator of the group it follows that all these groups belong to the given system.

Since a solvable group that does not belong to this system must either coincide with its group of cogredient isomorphisms, or one of its successive groups of cogredient isomorphisms must have this property, it seems desirable that a special study should be made of the solvable groups which coincide with their groups of cogredient isomorphisms. A large number of these groups may be represented as substitution groups of a low degree, for every solvable group which contains no operator besides identity that is commutative to each one of its operators is clearly a group of this kind.

During the last three years there has been developed a theoretically elegant method to prove the solvability of any given group. It also had its origin in Lie's groups and its

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* Miller, BULLETIN, vol. 4, pp. 510–515.

† Ahrens, *Leipziger Berichte*, vol. 49 (1897), pp. 616–626.

first developments were due to suggestions received from this great investigator. We proceed to give an outline of the method and to indicate some of the related unsolved problems.

If we multiply the inverse of every operator of a group G into all its transforms with respect to the operators of G we obtain a series of operators which form a characteristic* subgroup of G and this is the smallest self-conjugate subgroup of G with respect to which it is isomorphic to an Abelian group. This characteristic subgroup has been called the first derivative or the commutator subgroup of G. The first derivative of the first derivative is called the second derivative of G, etc.

If we arrive at identity by finding the successive derivatives of G it must be a solvable group.† *This condition is necessary as well as sufficient.* That it is necessary follows directly from the facts that a group which coincides with its first derivative cannot be isomorphic to any Abelian group and that every quotient group of a solvable group is solvable. That it is also sufficient is a consequence of the well known theorem that the factors of composition of G may be obtained by combining those of its first derivative and the prime factors of the order of the corresponding quotient group.

It is evident that the subgroup which corresponds to identity of the group of cogredient isomorphisms is an Abelian characteristic subgroup, and that that which corresponds to identity in any one of the successive groups of cogredient isomorphisms is characteristic but not always Abelian. Hence the series of the successive groups of cogredient isomorphisms and that of the successive derivatives have this marked difference, that in one series the quotient group is always Abelian while the corresponding characteristic subgroup does not necessarily have this property, and in the other series the reverse is true. Since this system includes all solvable groups it must include the preceding system.

A group which coincides with its first derivative is called a perfect group. From the above it follows that an unsolvable group is either perfect or one of its derivatives is perfect. Hence it is seen that the study of perfect groups is fundamental in the study of unsolvable groups. It is clear that every simple group of composite order is perfect and that the direct product of any number of such groups

*Frobenius, *Berliner Sitzungsberichte*, 1895, p. 183.
†Miller, *Quar. Jour. of Math.*, vol. 28 (1896), p. 266.

has the same property. The system of perfect groups, therefore, is larger than that of simple groups. Each of these two systems contains the same number of groups that can be represented as transitive substitution groups of a prime degree,* for a perfect transitive group of a prime degree is simple.

The well-known articles of Frobenius "Ueber auflösbare Gruppen,"† seem to the writer to be by far the most important direct contribution to the theory of solvable groups that has been made during the last five years. The theorem that every group whose order is not divisible by the square of a prime number is solvable is especially valuable and it is one of the most general known theorems in this line. The articles contain a number of other useful theorems. Most of these are found in Chapter XV. of Burnside's Theory of Groups. This excellent chapter contains also some new theorems by the author of the book.

With respect to the groups of a low order (and probably also with respect to those of a high order) solvability is the rule and unsolvability is the exception. This also applies to the groups which may be represented as substitution groups of a small degree. Hölder has recently enumerated all the unsolvable groups whose order does not exceed 479. The following table contains the orders of these twenty-five groups:‡

Order	60	120	168	180	240	300	336	360	420
Number	1	3	1	1	8	1	3	6	1

It is well known that the degree of a primitive solvable group is a power of a prime number. The following table§ gives the numbers of all these groups whose degree is less than 25:

Degree	3	4	5	7	8	9	11	13	16	17	19	23
Number	2	2	3	4	2	7	4	6	10	5	6	4

The following table gives the numbers of the insolvable groups which may be represented as substitution groups whose degree is less than 12:‖

Degree	5	6	7	8	9	10	11
Number	2	4	6	16	41	106	228

* Miller, *Amer. Jour. of Math.*, vol. 20 (1898), p. 277.
† Frobenius, *Berliner Sitzungsberichte*, 1893-95, pp. 337, 1027.
‡ Hölder, *Math. Annalen*, vol. 46 (1895), p. 420.
§ Miller, *Amer. Jour. of Math.*, vol. 20 (1898), p. 241.
‖ Ibid., p. 282.

§ 2. *Simple groups.*

We have already observed that there is an intimate relation between the questions of solvability and simplicity. In fact, if there were no simple groups of a composite order every group would be solvable. The simple groups also play a fundamental rôle in other domains of the theory of groups and it is a matter of the greatest interest that Killing and Cartan have recently succeeded in proving that there is no finite continuous simple group besides the five groups which have respectively 14, 52, 78, 133 and 248 parameters and the four great systems established by Lie.*

In the theory of discontinuous groups (finite and infinite) as well as in that of the infinite continuous groups the enumeration of all the possible simple groups seems to offer much greater difficulties. Although infinite systems of such groups are known, yet these do not give any assurance that a complete enumeration can ever be made, since the general theory of these groups is very complex. In what follows we shall confine our attention to the simple groups of a finite order.

The division of groups into simple and compound is due to Galois who also observed that there is no simple group of any composite order less than 60.† About six years ago Hölder proved that there are only two simple groups of a composite order less than 200, viz., the well-known groups of orders 60 and 168.‡ Soon after this Cole considered all the possible simple groups whose orders are composite and lie between 200 and 661. He found that there are only three such groups, viz., one for each of the orders 360, 504 and 660.§

The simplicity of the group of order 504 was discovered by Cole. This is the lowest order for which a simple group has been discovered within the last decade. The historical interest of this group has been greatly enhanced by the fact that it served as a starting point for some very interesting investigations by Moore which led to the proof of the existence of a doubly infinite system of simple groups. The given group may be represented as a transitive substitution group of degree 9 since it contains a subgroup of order 56. Kirkman gave it in this form in his list of the transitive groups whose degree does not exceed ten,‖ but he appar-

* Cf. Cartan, Thesis, Paris, 1894.

† Lettre de Galois à M. Chevalier, *Liouville's Journal*, vol. 11 (1846), p. 409.

‡ Hölder, *Math. Annalen*, vol. 40 (1892), pp. 55–88.

§ Cole, *Amer. Jour. of Math.*, vol. 15 (1893), pp. 303–315.

‖ Kirkman, *Proc. Lit. and Phil. Soc. of Manchester*, vol. 3 (1863), pp. 133–152.

ently did not know that it is simple.

Burnside has continued the search for simple groups of a low order by investigating the orders from 661 to 1092. He found only one simple group whose order lies in this region, viz., the well-known group of order 1092 which may be represented as a transitive group of degree 14. This belongs to the system of groups of the modular equation for the transformation of elliptic functions. If we remember that Cole found that the given group of order 504 is simple while seeking all the possible groups that can be represented as transitive substitution groups of degree 9*, we see that only negative results have been reached by the study of all the possible simple groups whose order is less than 1093. Regardless of this fact it seems desirable that the search of simple groups along this line should be continued somewhat further.

During the last few years very interesting systems of simple groups have been obtained by means of the application of an extension of the elementary concept of congruences, or the theory of the Galois field whose order is a power of a prime number. Moore proved a fundamental theorem in regard to the theory of this field, viz.: *Every existent field is the abstract form of a Galois field whose order is a power of a prime number*;† *i. e.*, every existent finite field of quantities such that the sum, difference, product and quotient (except by 0) shall belong again to the field must be a Galois field of order a power of a prime.

By the application of this theory Moore arrived at the doubly infinite system of simple groups which was mentioned above, and which is contained in the same article as the given theorem. A few months later Burnside was led to the same system by means of somewhat similar considerations.‡ Still further generalizations have led Dickson and Burnside to triply infinite systems of simple groups. The first of these systems was found by Dickson while working on his Chicago thesis.§ Since this time Dickson has arrived at other systems of simple groups‖ and he is continuing his important investigations in this direction.

It is an interesting fact that nearly all the known simple groups belong to one or more of the known infinite systems.

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*Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1892), pp. 253–4.

†Moore, Chicago Mathematical Papers, 1893, p. 211.

‡Burnside, *Proc. Lond. Math. Soc.*, vol. 25 (1894), pp. 113–139.

§Dickson, *Annals of Mathematics*, vol. 11 (1897).

‖Dickson, *Quar. Jour. of Math.*, vol. 29, pp. 169–178.

The groups of orders 7920 and 95040 whose minimum degrees when they are represented as transitive substitution groups are 11 and 12, respectively, seem to be exceptions.* This, however, is no index that they do not form a part of some such systems that have not yet been developed. The fact that there are isolated cases in the finite continuous groups might perhaps lead one to suspect that similar cases might occur among the groups of a finite order.

It is evident that we can always construct more than one group of a given composite order except when none of its prime factors diminished by 1 or 0 is divisible by another one of them. In this case (as well as in the case when the order is a prime number) we can construct only the cyclical group. As it is not possible to construct more than one simple group for each of the lowest orders for which such groups are possible, it was a question of considerable interest to inquire whether it is possible to construct two simple groups of the same order and what is the lowest order for which this is possible. The first part of this question has just been answered in the affirmative by Miss Schottenfels of Chicago University. She succeeded in proving that there is at least one simple group of order $8!\div 2$ which is not simply isomorphic to the alternating group of degree 8.

A search for simple groups has also been made among the groups whose orders involve a small number of prime factors. It is evident that every group whose order is a prime number is simple. In seeking all the groups whose orders are composite and contain no more than three prime factors Netto, Cole and Glover, and Hölder observed that none of these groups are simple. Soon after this Frobenius observed that the group of order 60 is the only simple group whose order is composite and contains no more than four prime factors. This is clearly equivalent to the proof that all the other groups whose orders contain no more than four prime factors are solvable, for an unsolvable group must either be isomorphic to a simple group of composite order (which may be the group itself) or it must contain a characteristic subgroup that has this property.

Burnside has carried investigations along this line somewhat farther in his "Notes" published in the *Proceedings of the London Mathematical Society*, 1894–5. He proved, among other things, the correctness of a remark by Frobenius to the effect that there are only three simple groups whose

*BULLETIN, vol. 4 (1898), pp. 382–389.

order is the product of five prime numbers, viz., one group of each of the orders 168, 660, and 1092. The possible simple groups whose orders are products of six or a larger number of prime factors have not yet been determined. While the difficulty of this problem increases rapidly with the number of the prime factors, yet it seems quite desirable that investigations along this line should be continued.

We observed above that the study of congruences has led to remarkable systems of simple groups. This study has been by far the most helpful in determining such groups. The study of substitutions has led to a number of individual simple groups as well as to the well known system of alternating groups whose degree exceeds 4. Within the last year Beke has published a new proof of the simplicity of the groups of this system.* This proof is based upon very elementary concepts. Since every self-conjugate subgroup of a transitive group of a prime degree must include all its substitutions of a prime order, these substitutions must always generate a simple group, which is evidently of a composite order whenever the group contains more than one subgroup of order p. Hence every transitive group of a prime degree that is not included in the metacyclic group of this degree contains a simple group of composite order as self-conjugate subgroup. This may be the group itself. The quotient group with respect to this simple group is always cyclical, *i. e.*, such a group contains only one composite factor of composition.

While the proof of the existence of simple groups is of great interest, yet it leaves many interesting questions in regard to these groups unsolved. One of the most important of these is the determination of their subgroups. In fact, it might be said that the entire theory of the groups of a finite order is involved in the subgroups of the system of simple groups which is composed of the alternating groups; for every group whose order does not exceed n is clearly a subgroup of the alternating group of degree $n + 2$.

In a paper read at the last summer meeting of the AMERICAN MATHEMATICAL SOCIETY, Moore gave a determination of all the subgroups of the generalized modular group of a finite order. This is an extension of the noted paper by Gierster† in which he determined all the subgroups of the modular group and found that it can contain only three kinds of subgroups besides the cyclic and the metacyclic ones. The generalized modular group was first studied by Mathieu

* Beke, *Math. Annalen*, vol. 49 (1897), p. 581.
†Gierster, *Math. Annalen*, vol. 18 (1881), pp. 319-365.

with regard to its cyclical subgroups. In an earlier paper Moore has proved that it is simple except for a few special cases,* and in the present paper he gives a complete determination of its subgroups. He has kindly favored me with the following important statements in regard to this group:

The modular group Γ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

of the complex variable ω, where α, β, γ, δ are rational integers, has for every rational prime q a self-conjugate subgroup $\Gamma_{\mu(q)}$ of finite index $\mu(q)$ containing all substitutions $(\alpha, \beta, \gamma, \delta)$ for which $\alpha \equiv 1$, $\beta \equiv 0$, $\gamma \equiv 0$, $\delta \equiv 1 \pmod q$. The corresponding quotient group is conveniently given as the finite modular group $G^{q+1}_{\mu(q)}$ of substitutions $(\alpha, \beta, \gamma, \delta)$ on the $q+1$ marks, $\infty, 0, 1, \cdots, q-1$, where the $\alpha, \beta, \gamma, \delta$ are integers taken modulo q. By generalizing from the Galois field of rank 1 to that of rank n we have the generalized finite modular group $G^{q^n+1}_{M(q^n)}$ of order $M(q^n) = q^n(q^{2n}-1)$ or $q^n(q^{2n}-1)/2$. Its subgroups are of three kinds: (1) metacylic or solvable groups, (2) and (3) groups of the abstract character of certain groups $G^{q^n+1}_{M(q^n)}$, or of certain groups $G^{q^n+1}_{\frac{1}{2}M(q^n)}$, $q > 2$; the latter group being obtained by extending the former by the substitution $\omega' = \rho\omega$, where ρ is a primitive root of the Galois field of order q^n.

Thus the doubly infinite system of simple groups $G^{q^n+1}_{M(q}$ $(q^n + 2^1, 3^1)$ determines by the decomposition of the subgroups of its constituent groups, apart from the simple groups of prime order, only simple groups of the original system. An equation of degree q^n+1 $(q^n + 2^1, 3^1)$ whose Galois group is the $G^{q^n+1}_{M(q^n)}$ has resolvents of degree D, $D < q^n + 1$, only in the cases $q^n = 5^1, 7^1, 11^1, 3^2$, when D is respectively $5, 7, 11, 6$. For the case $n = 1$, this is a noted theorem of Galois and Gierster.

The group $\Gamma[\Omega]$ of all unimodular substitutions $(\alpha, \beta, \gamma, \delta)$ where the coefficients are algebraic integers of an algebraic field Ω[Körper Ω] has, for every prime functional Π [Weber] of absolute norm q^n, a self-conjugate subgroup $\Gamma_{M(q^n)}[\Omega]$. The quotient group is the group $G^{q^n+1}_{Mq^{(n)}}$.

*Moore, Chicago Mathematical Congress Papers, 1893, pp. 208–242. This is the system of simple groups which was mentioned above, p. 233.

The substitutions of the modular group Γ leave invariant a modular function $J(\omega)$. The transcendental modular equation $J = J(\omega)$ has algebraic resolvents, modular equations, with Galois groups $G^{\psi+1}_{M(\psi)=\mu(\psi)}$. The group $\Gamma[\Omega]$ for a field Ω of degree > 1 is improperly discontinuous and so has no corresponding automorphic function. But if an automorphic function $K(\omega)$ belongs to a group whose substitutions belong to an algebraic field Ω, then the generalized modular equation $K = K(\omega)$ has algebraic resolvents whose Galois groups are certain generalized finite modular groups $G^{\psi^n+1}_{M(\psi^n)}$ or certain of their subgroups. This remark connects the present paper with recent work of Fricke and Bianchi on certain special classes of automorphic functions (*cf.* Klein-Fricke, Automorphe Functionen, vol. 1, 1897, p. 586 fg.).

§ 3. *Substitution groups.*

In what might probably be called the earliest work on groups, viz., "Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebrica delle equazioni generali di grado superiore al quarto," di Paolo Ruffini, Bologna, 1799, the substitution groups are divided into transitive and intransitive as well as into primitive and imprimitive groups. These divisions have been adopted by all the subsequent writers. The primitive groups present the greatest difficulties and they have therefore received most attention.

One of the most interesting problems in regard to the primitive groups which do not include the alternating group is the determination of the smallest degree of any of its substitutions that differ from identity. Jordan has called this minimum degree the class of the primitive group and he was the first to make important investigations along this line.* In fact, Jordan's work and influence in this direction have been so preëminent that it would seem proper to call the given problem Jordan's problem.

During the last few years Bochert, Jordan, and Maillet have made important progress towards the solution of this problem. The work of Bochert is contained in a series of articles published in the *Mathematische Annalen*.† These articles are based upon very elementary principles. One of the most important of these is a direct consequence of the commutator of two substitutions and it may be stated as

* Cf. Jordan, *Liouville's Journal*, vol. 16 (1871).

† Vols. 29, 33, 40, 49.

follows: *If a group contains two substitutions that are not commutative it must also contain a positive substitution whose degree does not exceed three times the number of common elements of the two given substitutions.*

A special case of this principle, which can frequently be employed, is the fact that the commutator of two substitutions which have only one common element is always a substitution of degree 3. While Jordan, Sylow, and Bochert employed the commutator of two substitutions as an instrument, yet none of them seems to have made a special study of the properties of this operator. We have seen that this operator has some applications in substitution groups besides those which relate directly to the abstract group properties and hence it seems very desirable that its properties should be more completely determined. For instance, it would be very desirable to find a method by means of which the commutator of two substitutions could be obtained almost as readily as their transforms with respect to each other,* or a method by means of which we can easily see the form of the commutator whenever the substitutions are given. It would also be interesting to find out whether the commutators of the operators of a group constitute all the operators of the subgroup which they generate; *i. e.*, whether all the operators of the first derivative are commutators of operators in the group.

While the commutator of two substitutions is more complex than the transform of one of them with respect to the other, yet it sometimes has a decided advantage on account of its more nearly symmetric form, the interchange of the two substitutions giving merely the inverse of the commutator. If we permute the four operators or substitutions of any commutator in every possible manner we obtain eight commutators which may be distinct operators;† each of the other 16 forms is identically equal to unity, hence none of these 16 forms is a commutator. The transitive group of degree 4 and of order 8 is therefore the largest group of this degree that transforms a commutator into a commutator by merely permuting its operators.

The most practical theorem that Bochert found in the given series of articles may be stated as follows: *If a group of degree n does not include the alternating group and is more than simply transitive its class exceeds $\frac{1}{2}n - 1$, if it is more than doubly transitive its class exceeds $\frac{1}{3}n - 1$, and if it is more than*

* Jordan, Traité des substitutions, 1870, p. 23.

† Cf. Bochert, *Math. Annalen*, vol. 40 (1892), p. 159.

triply transitive its class is not less than $\frac{1}{2}n - 1$.* Jordan regarded this theorem as the most important advance made since the theorem of Sylow was published.† It should, however, be observed that this theorem differs widely from that of Sylow in regard to applications; while Sylow's theorem applies to groups regardless of the notation, that of Bochert is directly applicable only to primitive substitution groups. In a recent article published in *Liouville* ‡ Jordan succeeded in obtaining somewhat closer limits for the class than those that had been obtained by Bochert.

Maillet has published a large number of articles in which the class of a primitive group is considered. His memoir§ entitled, "Recherches sur la classe et l'ordre des groupes de substitutions" was crowned in 1896 by the Paris Academy with the Grand prix des sciences mathématiques. In the first part of this memoir the author treats the class of primitive groups which are isomorphic to the symmetric or the alternating group. In the second part he considers the highest order of a primitive group that does not include the alternating group. Both of these questions are old and they have been the subjects of a large number of memoirs. Maillet attacked them from new standpoints and succeeded in finding new and interesting results.||

It is well known that the metacyclic group of degree p, p being any prime number, is of class $p - 1$. The question naturally arises in what other primitive groups is the class equal to the degree diminished by unity. Maillet has devoted a great deal of attention to this question and he succeeded in proving that when the degree of a primitive group is less than 202 its class cannot be obtained by diminishing the degree by unity unless the degree is a power of a prime number.¶ In his thesis he also considered the primitive groups of degree n and classes $n - 2$ and $n - 3$**.

The question of the class of a primitive group is very closely related to that of its limit of transitivity and these two questions have frequently been studied together. In Vol. I of the *Bulletin de la Société Mathématique de France* Jordan published several elegant theorems on the limit of transitivity. It is a singular fact that these theorems have not found a place in the text books, especially the one which

* Bochert, *Math. Annalen*, vol. 40 (1892), p. 179.
† Jordan, *Liouville's Journal*, fifth series, vol. 1 (1895), p. 35.
‡ *Ibid.*
§ *Mém. sav. étr.*, vol. 32, no. 8.
|| *Cf.* Jordan, *Comptes rendus*, vol. 123 (1896), pp. 1107–1109.
¶ Maillet, *Bull. de la Soc. Math. de France*, vol. 25 (1897), pp. 16–32.
** Maillet, Thesis, *Paris*, 1894.

asserts that *a group of degree $p+k$, p being any prime number, cannot be more than k times transitive whenever $k>2$.* Quite recently the writer has given a somewhat different proof to two of these theorems and he has extended one of them. This proof is based upon the lemma (which does not appear to have been noticed) that *the quotient group of any transitive group of a prime degree with respect to any self-conjugate subgroup (except identity) is cyclical.**

The determination of all the groups of degree n has always been a prominent goal in the theory of substitutions and it has been called the principal object of the theory of substitutions.† Although this problem has been attacked by a large number of mathematicians yet it still remains far from a complete solution. For small degrees the problem can be solved very easily but the difficulty increases rapidly with the degree. We proceed to give a short account of recent progress in this direction.

In Vol. 24 of the *Quarterly Journal of Mathematics*, Askwith published two articles in which he investigated the groups whose degree does not exceed eight. In the following volume of this journal Cayley gave the results of Askwith together with some additions. Numerous omissions occur in this list, even with respect to groups that were published at an earlier date. A supplementary list together with some interesting methods was published by Cole about a year later.‡ This was followed by numerous additional articles § by Cole and the writer in which the intransitive, the imprimitive, and the primitive groups are given up to degrees 11, 15, and 17 respectively. In several of these papers there is no attempt to give anything besides the results.

In these enumerations it frequently happens that the same abstract group is given more than once. This does not cause any inconvenience in regard to the transitive groups, for if we know the properties of all the subgroups of a given group we know definitely in how many ways it may be represented as a transitive substitution group. Hence the determination of all the possible transitive groups of a given order is equivalent to the determination of all the abstract groups of this order and the study of some of the properties of these groups. When a group is simple

* Miller, BULLETIN, vol. 4 (1898), p. 141.

† Serret, Algèbre supérieure, 5th edition, vol. 2, p. 283.

‡ Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1893), p. 184.

§ These articles have appeared in recent numbers of the BULLETIN, *Quar. Jour. of Math.*, *Proc. Lond. Math. Soc.*, and *Amer. Jour. of Math.*

the determination of all its different sets of subgroups such that each set includes all those that may correspond in a simple isomorphism of the group to itself is equivalent to the determination of all the different transitive forms in which it may be represented as a transitive substitution group.

There is no such simple relation between the abstract groups and the ways in which they can be represented as intransitive substitution groups. Very little has been done towards giving formulas by means of which all the possible intransitive substitution groups of a given order and any arbitrary degree may readily be determined. Such formulas have recently been published for the cases when the order is 4 or when it is the product of two unequal prime numbers.* The next step in this direction will probably be a formula by means of which we can directly ascertain the number of the intransitive groups whose order is the square of a prime and whose degree is an arbitrary number. It is evident that the total number of intransitive groups of any given order is infinite, while that of transitive groups is finite whenever the order is finite.

While a large number of the substitution groups have been found by tentative methods yet it is very easy to establish general theorems by means of which all trials may be avoided, at least, when the degree is small. It seems, however, that the number of the necessary general theorems increases rapidly with the degree. The known theorems are sufficient to determine all the possible substitution groups at least as far as degree nine without any trials. The present theory of the imprimitive groups is perhaps more unsatisfactory than that of the other two divisions of substitution groups.

It is frequently desirable to express substitutions analytically. In his Chicago thesis of 1896, Dickson has considered (among other things) the determination of all quantics up to as high a degree as practicable which are suitable to represent substitutions on p^n letters, p being any prime number, as well as the determination of special quantities suitable on p^n letters where for each quantic the combination (p, n) takes infinitely many values. He established a number of interesting theorems and added very materially to our knowledge in regard to this subject.

* Miller, BULLETIN, vol. 2 (1896), pp. 332–334.

§4. *Abstract groups.*

Cayley called attention to the fact that a group is defined by means of the laws of combination of its symbols.* Such a group is called an abstract or operation group. In other words, all simply isomorphic groups are merely different representations of the same abstract group. Every concrete group (such as a substitution group) represents some abstract group and every abstract group may be represented by many concrete groups. All the abstract groups might, therefore, be studied by means of a system of concrete groups selected in such a manner that one, and only one, belongs to each set of simply isomorphic groups. The regular substitution groups, as is well known, form such a system.

This method of study would have the disadvantage that in concrete groups the generational relations are not always defined with clearness, *i. e.*, the concrete generators sometimes involve conditions which do not stand out prominently in the concrete group. Hence it is frequently desirable to study abstract groups without any reference to any particular representation. This requires much greater care as to definition than is necessary for concrete groups, *e. g.*, any set of substitutions, such that no two of them are identical, obeys all the conditions of a group except the condition that the product of any two of them, or the square of any one, should be contained in the set. Hence the definition of a substitution group does not need to contain any other condition and it is therefore exceedingly simple.

During recent years accurate definitions of an abstract group have been formulated. These are principally due to Frobenius and Weber. Such definitions are stated at the beginning of the second volume of Weber's Algebra. It is to be hoped that practical uniformity in regard to the definition of an abstract group may be attained and that the energies of a large number of investigators may thereby be directed towards the same difficulties. While the term abstract group is frequently defined very inadequately, even in recent works, yet this deficiency is generally supplied by the inherent properties of the concrete operators of the groups which are actually used.†

Although a clear insight into the generational relations of a concrete group does not always occupy a prominent place in the study of these groups, yet it is an important part of

* Cayley, *Amer. Jour. of Math.*, vol. 1 (1878), p. 51.

† Cf. Lie, *Le Centenaire de l'École Normale*, 1895, p. 485.

their study. Some important group properties follow directly from these relations; *e. g.*, if the generators of a group satisfy all the generational relations of another and can be placed in 1, 1 correspondence with them the first of these groups is a quotient group of the second.* We thus arrive directly not only at self-conjugate subgroups of the given group but also at the corresponding quotient groups.

Hence we observe the importance of finding the generational relations of known systems of concrete groups. This seems to offer a desirable field of investigations. Quite recently these relations were determined for the symmetric and the alternating group.† For the Abelian groups they have been known for a long time and for the Hamiltonian groups they follow directly from the known properties of these groups. When the order of a group is the product of a small number of prime factors the generational relations were determined at the same time as the possible groups of these orders were determined.‡ In fact, the generational relations furnish one of the most satisfactory means of determining all the groups whose orders involve a small number of prime factors, while they do not appear to furnish the most direct means to determine all the groups which may be represented as substitution groups of a low degree.

The groups of order p^α, p being any prime number, are of especial interest since every group whose order is divisible by p^α must contain at least one subgroup of this order. Any progress in regard to the theory of these groups is therefore of more than ordinary importance. Among the most important theorems in regard to these groups are the following: Every group of order p^α is solvable and it contains a self-conjugate subgroup of order p^β, whenever $\beta < \alpha$. A subgroup whose order is less than p^α is self-conjugate in a subgroup which includes operators that are not contained in the former of these subgroups. Every operator of the group is commutative to at least p operators of the group.

The following general theorem was published during last year and it includes some other well known theorems in regard to these groups:§ *A group of order p^α contains a subgroup of order $p^{\alpha-\frac{\beta(\beta-1)}{2}}$, such that each of the operators of this*

* Moore, *Proc. Lond. Math. Soc.*, vol. 28 (1896), p. 359.
†Ibid.
‡ Cf. Hölder, *Math Annalen*, vol. 43 (1893), pp. 335-360.
§Miller, *Messenger of Mathematics*, vol. 27 (1897), p. 120.

subgroup is commutative to every one of the operators of a subgroup of order p^s that is contained in the given subgroup. As an important corollary we have: *A group whose order exceeds $p^{\frac{n(n-1)}{2}}$ contains a commutative subgroup of order p^n.* If we let $n = 2$ we have the theorem that a group whose order is p^2 must be Abelian and if $n = 3$ we have that every group whose order is p^4 contains an Abelian subgroup of order p^3.

In the early development of the theory of groups a large number of the theorems were proved by means of substitution groups. It soon became desirable to give proofs which are independent of any particular notation. One of the first important steps in this direction was taken by Frobenius in his proof of Sylow's theorem.* In a more recent paper the same author has proved the important theorem that every simple isomorphism of an abstract group † to itself can be obtained by transforming the group. This theorem is equally true of regular substitution groups but it is not true in regard to all substitution groups, *e. g.*, the modular groups of degrees 7 and 11 are known to have simple isomorphisms to themselves which cannot be obtained by transforming the substitution groups.

While it is a matter of interest to give proofs which are independent of any special notation yet it does not appear desirable to do this at the expense of clearness, *e. g.*, if we make a regular group simply isomorphic to itself we may suppose that all the substitutions begin with the same element. The second elements of the corresponding substitutions give then the substitution which transforms the given regular group into the given isomorphism to itself. Hence we observe that if a regular group of degree n is transformed into itself by the largest possible group of this degree, the subgroup of this group which contains all its substitutions that do not contain a given element is simply isomorphic to the group of isomorphisms of the given regular group. It seems that some questions in regard to the group of isomorphisms can be studied most conveniently by means of this subgroup. The most elementary concept of the group of isomorphisms is that of a substitution group whose elements correspond to the operators of the group which are not characteristic, *i. e.*, to those operators which do not correspond to themselves in every simple isomorphism of the group to itself. It is evident that identity is a characteristic operator of every group. If a group is Abelian and

* Frobenius, *Crelle*, vol. 100 (1887), p. 179.
† Frobenius, *Berliner Sitzungsberichte*, 1895, p. 184.

contains only operators of order p in addition to unity, its group of isomorphisms regarded as such a substitution group is transitive and there is only one characteristic operator.* In all other cases this group of isomorphisms is either intransitive or there is more than one characteristic operator, *e. g.*, the group of isomorphisms of the quaternion group is transitive but there are two characteristic operators.

While the group of isomorphisms may be regarded as a substitution group yet it seems to be more commonly regarded as the abstract group which is simply isomorphic to the given substitution group. As we are chiefly concerned with the group properties of the group of isomorphisms it may be desirable to emphasize this fact by calling it an abstract group. This also avoids the trouble of selecting any particular notation. If we remember that the abstract group properties should form the most prominent part of a substitution group, and that the particular notation should tend to deepen our insight into the abstract group rather than towards making it more superficial, it may not appear essential to call the group of isomorphisms an abstract group.

It is frequently desirable to know all the possible abstract groups of a given order. We have already observed that all the possible groups of any order that do not involve more than three prime factors are known. Only special cases have been investigated when the order involves more than three prime factors. The most important of these has recently been considered by Hölder, who succeeded in finding all the groups whose orders are not divisible by the square of any prime number.† His investigations are based upon results obtained by Frobenius when he proved that all these groups are solvable.

The other extreme case, viz., when the order of the groups is a power of a single prime number, seems to present much greater difficulties. Young and Hölder found at almost the same time all the groups whose order is the fourth power of a prime. Quite recently Bagnera has considered all the possible groups whose order is the fifth power of a prime number.‡ The difficulties of this problem seem to increase very rapidly with the index of the power of the prime number.

The results of this article by Bagnera do not appear to be entirely correct, at least as far as they relate to the groups of order 32. The number of these groups that are Abelian

* Moore, BULLETIN, vol. 2 (1895), p. 33.
† Hölder, *Göttinger Nachrichten*, 1895, pp. 211–229.
‡ Bagnera, *Annali di Matematica*, 1898, pp. 137–228.

or contain a subgroup of order 8 whose operators are self-conjugate in the entire group is correctly stated, but the numbers of those that contain only 2 or 4 operators that are self-conjugate in the entire group are given incorrectly, the number of the former being only 10 instead of 12, and of the latter 19 instead of 16. As there are evidently 7 Abelian groups of this order and 15 that contain 8 operators which are commutative to every operator of the group, the total number of these groups is 51.

Among the much more special cases that have been considered are all the groups whose order does not exceed a given number. This problem has recently been solved for all the orders less than 64, but the results for the orders which exceed 47 have not yet been published. In regard to the groups of order 32, Bagnera's recent article fixes the number of the possible groups of this order at 50 while the actual number seems to be 51 as given above. Levavasseur had supposed their number to be beyond 75.* When the number of the possible groups is so large the probability of making an error is much greater than when the number is small and it is therefore very desirable that such questions should be investigated by independent methods.

A few other very special cases of the general problem of finding all the groups whose order is the product of four prime numbers have been incidentally considered, viz., the orders $8p$† and $2p^3$. These cases are, however, too special to have much influence on the general problem. The complete solution of this problem would be a matter of considerable interest and it seems to be the next desirable step in this line of work.

Conclusion.

"In recent years many groups of permutations of 6, 7, 8, 9, ··· letters have been made known. The problem would be to determine in each case the minimum number of variables with which isomorphic groups of linear substitutions can be found. Secondly, I want to call your particular attention to the case of the general equation of the eighth degree. I have not been able in this case to find a material simplification, so that it would seem as if the equation of the eighth degree were its own normal problem. It would no doubt be interesting to obtain certainty on this point."‡

The first of these problems seems to be still a standing

* Levavasseur, *Comptes rendus*, vol. 122 (Jan., 1896); cf. Miller, *Comptes rendus*, Feb., 1896.

† Miller, *Phil. Magazine*, vol. 42 (1896), p. 195.

‡ Klein, The Evanston Colloquium, 1894, p. 74.

problem while the second has recently been solved by Wiman, who proved more than the given problem by showing that there is no material simplification for the general equation (the symmetric and the alternating substitution group) whose degree exceeds 7.* It may be remarked in passing that Wiman's footnote on page 58 is quite erroneous, the doubly transitive group of order 56 and degree 8 being found in the much earlier list by Kirkman† while the triply transitive group of order 1344 and degree 8 is found in a number of different places in addition to Kirkman's list.

The linear groups are of extreme importance on account of their numerous direct applications. Every group of a finite order can clearly be represented in many ways as a linear substitution group since the ordinary substitution (permutation) groups are merely very special cases of the linear groups. The general question of representing such a group with the least number of variables seems to be far from a complete solution. It is closely related to that of determining all the linear groups of a finite order that can be represented with a small number of variables.

Klein was the first to determine all the finite binary groups‡ while the ternary ones were considered independently by Jordan§ and Valentiner.‖ The latter discovered the important group of order 360 which was omitted by Jordan and has recently been proved simply isomorphic to the alternating group of degree 6.¶ Maschke has considered many quaternary groups and established, in particular, a complete form system of the quaternary group of 51840 linear substitutions.**

Frobenius has recently published an article on the representation of finite groups by linear substitutions, which contains a large number of new results.†† This is a continuation of his article on the prime factors of the group determinant.‡‡ In the latter article the author has associated with every finite group H of order h a matrix of degree h whose elements are dependent upon h variables. In the

* Wiman, *Göttinger Nachrichten*, 1897, pp. 53 and 191.

† Kirkman, *Proc. Lit. and Phil. Soc. of Manchester*, vol. 3 (1863), pp. 133–152.

‡ Klein, *Math. Annalen*, vol. 9 (1875), p. 183.

§ Jordan, *Atti delle Reale Accademia di Napoli*, 1880.

‖ Valentiner, "De endelige Transformations-Gruppers Theori," 1889.

¶ Wiman, *Math. Annalen*, vol. 47 (1896), pp. 531–556.

** Maschke, *Math. Annalen*, vol. 33 (1889), pp. 317–344; cf. articles by the same author in vols. 30 and 36.

†† Frobenius, *Berliner Sitzungsberichte* (1897), pp. 994–1015.

‡‡ Ibid., 1896.

present article he points out, among other things, the connection between this matrix and the linear substitutions by means of which H and its isomorphic groups may be represented.

In a recent number of *Liouville* Laurent published an "Exposé d'une théorie nouvelle des substitutions,"* in which he proposes to create an algorithm by means of which the general theory of substitutions may be presented. He employs not only the product and powers of substitutions but also their sums and differences. The term *new theory* should perhaps be understood to mean that the author is dealing mostly with facts that have not appeared in treatises. References would seem to have made the article more useful. The author adds: "Ceux qui voudront bien lire les pages qui suivent se convaincront que je n'ai fait qu'effleurer un sujet très vaste."

In closing we would repeat what was stated at the beginning of this report that we have aimed to call attention to only a few of the important recent advances in the theory of groups. In almost all parts of higher mathematics the group theory is continually taking a more prominent position† and it would require a man of riper years and much wider attainments than those possessed by the writer to give a harmonious and extensive account of the marvelous recent progress in this field. If our humble efforts shall be of service to some beginner in leading him to problems whose solution will assist him to penetrate the rich fields of this theory they will be amply rewarded.

CORNELL UNIVERSITY,
August, 1898.

NOTE ON BURNSIDE'S THEORY OF GROUPS.

BY DR. G. A. MILLER.

IT is well known that Professor Cayley published an enumeration of the possible substitution groups whose degree does not exceed eight‡ and that Professor Cole pub-

* Laurent, *Liouville's Journal*, vol. 4 (1898), pp. 75–119; cf. recent articles by the same author in *Nouvelles Annales*.

† Klein-Fricke, Vorlesungen über die Theorie der automorphen Functionen, 1897, p. 1.

‡ Cayley, *Quar. Jour. of Math.*, vol. 25 (1891), pp. 71 and 137.

lished a large supplementary list of these groups.* Under the guidance of the latter the writer examined all these groups and observed that an intransitive group of degree 8 and order 16 should be omitted from these lists and that two transitive groups of the same degree and of orders 24 and 1344 respectively should be added.† No additional errors have been published.

In the recent work of Burnside some of these groups are employed for the sake of illustrations. It seems desirable to explain the three instances where the results of Burnside differ from those of the given lists since this may assist the reader of his valuable work. These differences occur in the enumeration of the imprimitive groups of degree 6, the intransitive ones of degree 7, and the primitive ones of degree 8. We proceed to consider them in order.

While seeking all the possible imprimitive groups of degree 6 our author speaks (p. 182, l. 8) of an imprimitive group of degree 6 and order 12 that contains operators of order 4. It is easy to prove that such a group cannot be constructed. It would have to contain a subgroup of order 6 since it contains two systems of imprimitivity. Hence it would contain only one subgroup of order 3, and, as it could not be Abelian, it would contain 3 subgroups of order 4. Its operators would have to transform these three subgroups according to the symmetric group of degree 3; *i. e.*, the group could contain only one subgroup of order 2 and could therefore not be represented as a transitive group of degree 6.

By omitting "or IV" on p. 182, l. 8 and writing seven in place of eight in the following line the results of our author in regard to the imprimitive groups of degree 6 agree with the older lists and we have just proved that these changes should be made. In seeking the possible intransitive groups of degree 7 which contain 4, 3 as their systems of intransitivity, pp. 163 and 164, our author evidently includes a case in which the constituent of degree 4 is intransitive. It is therefore necessary to omit paragraph VI near the bottom of p. 164 and to write 18 in place of 22 in the first line of p. 165.

Among the primitive groups of degree 8 those containing the Abelian group of order 8 which includes 7 operators of order 2 are of special interest since they include all the solv-

*Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1893), p. 184.

†Miller, *Bulletin of the New York Mathematical Society*, vol. 3 (1894), p. 242.

able primitive groups of degree 8.* As the group of isomorphisms of this Abelian group is the simple group of order 168† all the groups in question must be subgroups of the triply transitive G^8_{1344} and they must correspond to subgroups of the given group of isomorphisms.

The substitutions of G^8_{1344} that do not contain a given element constitute a G^7_{168} and may conveniently be regarded as representing the given group of isomorphisms. To the two transitive types of subgroups in G^7_{168} there must correspond doubly transitive subgroups of G^8_{1344}. We thus obtain the two solvable primitive groups of degree eight, viz., G^8_{56} and G^8_{168}.

If an intransitive subgroup of G^7_{168} were contained in a primitive group it would be of degree 7 and contain one transitive constituent of degree 3 and one of degree 4. As such an intransitive group could not include any substitution whose degree is less than 4 it would have to include the transitive four group. This is clearly impossible.‡

Hence G^8_{56} and G^8_{168} are the only solvable primitive groups of degree 8 and the two other groups given by our author, p. 210, l. 29, whose orders are $2^5 3$ and $2^4 3$ respectively, are not primitive. Lines 28 and 29 on p. 211 which relate to the same groups should therefore be omitted. The differences between the lists of our author and the earlier lists mentioned above are therefore not due to any error in the latter. It is necessary to add that while we considered these groups in our recent Chicago seminar two of the members, Mr. Grant and Miss Schottenfels, found the first two errors mentioned above.

CORNELL UNIVERSITY,
September, 1898.

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* Jordan, Traité des substitutions (1870), p. 398.

† Moore, BULLETIN, vol. 1 (1894), p. 61.

‡ Netto, Theory of substitutions (Cole's edition, 1892), p. 95.

ON A REGULAR CONFIGURATION OF TEN LINE PAIRS CONJUGATE AS TO A QUADRIC.

BY PROFESSOR F. MORLEY.

(Read before the American Mathematical Society at the Meeting of October 29, 1898.)

At the meeting of the London Mathematical Society, June, 1898, I showed a model of ten lines in space, each intersecting three others perpendicularly.* Subsequently Professor Study, seeing the model, remarked that there should be an analogous configuration of ten pairs of lines conjugate as to a quadric. This new configuration can be established as follows:

(1) Consider a ruled quadric H, and denote by Q an inscribed quadrilateral formed by two right generators and two left generators. The diagonals of Q are a pair of conjugate lines, say P. Thus Q determines P.

(2) Call two pair of generators harmonic when they cut another generator in harmonic pairs of points. Then two harmonic pairs of right generators will cut any conic of the surface H at the ends of conjugate chords, and the left generators through these ends will also be harmonic pairs. Hence calling two inscribed quadrilaterals Q and $\overline{Q}$ harmonic when their right generators and also their left generators are harmonic pairs, we can say that *harmonic quadrilaterals determine meeting conjugate pairs*, where we mean that each of the one pair meets each of the other.

(3) Now two quadrilaterals Q and Q' have a common harmonic quadrilateral; therefore two conjugate pairs are met by a third conjugate pair; that is, the two lines which met the four are themselves conjugate.†

(4) Take now three conjugate pairs P_1, P_2, P_3. Let the pair meeting both P_2 and P_3 be P_1'; thus we have three new pairs P_1', P_2', P_3'. Let the pair meeting P_1 and P_1' be P_1''; thus we have three new pairs P_1'', P_2'', P_3''. It is to be proved that these are met by one conjugate pair.

Replace the pairs by inscribed quadrilaterals, and the matter being merely one of harmonic constructions consider

* See *Proc. Lond. Math. Soc.* vol. 29.

† We set aside the special case in which the two pairs are met by an infinity of lines, which arises when the two quadrilaterals have the same right (or left) generators.

separately the right and left generators. Refer the pairs of right generators to the pairs of points where they meet a conic of the surface.

Then we are to prove that if p_1, p_2, p_3 are pairs of points on a conic, and the pair harmonic with p_2 and p_3 be p_1', and the pair harmonic with p_1 and p_1' be p_1'', then p_1'', p_2'', p_3'' are in involution. In other words, replacing the pairs of points by the lines through them, we are to prove that a triangle and its polar triangle as to a conic are in perspective, a known elementary proposition. The ten pairs of points on the conic are in fact the points in which the lines of the configuration 10_3, the intersection of a plane with a complete five point cut the conic to which it is self polar.

Thus as three pairs of points on a conic determine the configuration of ten pairs of points, so three quadrilaterals in the surface determine a configuration of ten quadrilaterals, and replacing each quadrilateral by its diagonal pair we have the configuration of Professor Study.

HAVERFORD COLLEGE,
November, 1898.

BESSEL'S FUNCTIONS.

Einleitung in die Theorie der Bessel'schen Funktionen, von Professor J. H. GRAF und Dr. E. GUBLER. Erstes Heft: Funktionen erster Art. Bern, 1898, Wyss & Co.

IN the last few years the study of Bessel's functions has been enriched by many contributions, the subject generalized for the complex variable, and the premises much more carefully defined.

Among English speaking investigators the interest seems to lie chiefly in two directions, theoretical consideration of the differential equation, and practical applications, numerical tabulation, etc.

The book under review does not favor any particular side but may be said to put the most emphasis upon the definite integral. Numerical tabulation is not touched upon in the part which has appeared; the remaining part, soon to appear, may have a chapter devoted to this purpose.

The first feature is a carefully arranged historic introduction, which contains an extensive list of books and memoirs

on the subject. The origin of the Bessel differential equation is ascribed to Riccati; after briefly citing various physical applications the history of the development of Kepler's problem is given from the time of its origin to the definition, by Bessel, of $\overset{\lambda}{J}(K)$, viz.:

$$2\pi\overset{\lambda}{J}(K) = \int_0^{2\pi} \cos(he - K\sin e)de.$$

The problemi tself is then stated and solved "according to Bessel," though the authors have skillfully modernized the solution by regarding the variable as complex throughout, determining coefficients by repeated use of Cauchy's theorem, and plotting multiple valued functions on a Riemann's surface. The chapter is closed by expressing these functions as definite integrals and infinite series, the latter leading up to the formal definition

$$\overset{a}{J}(x) = \sum_{\lambda=0}^{\infty} (-1)^{\lambda} \frac{\left(\frac{x}{2}\right)^{a+2\lambda}}{\lambda!\,\Gamma(a+\lambda+1)} \tag{1}$$

with which the treatment really begins.

The evaluations of $\overset{\frac{1}{2}}{J}(x)$, $\overset{-\frac{1}{2}}{J}(x)$ are first obtained, then the relations which exist when a is an integer n,

$$\overset{-n}{J}(x) = (-1)^n \overset{n}{J}(x), \quad \overset{n}{J}(-x) = \overset{-n}{J}(x), \quad \overset{-n}{J}(-x) = \overset{n}{J}(x). \tag{2}$$

By means of differential operators the usual relations are easily established; from the form

$$\overset{a}{J}(x) = (a-1)\frac{2}{x}\overset{a-1}{J}(x) - \overset{a-2}{(Jx)}$$

the general form

$$\overset{a}{J}(x) = \overset{a-m}{A_m J}(x) - \overset{a-m-1}{A_{m-1} J}(x)$$

is neatly proved by induction, wherein

$$A_m = \sum_{\lambda=0}^{E\left(\frac{m+1}{2}\right)} (-1)^{\lambda} \binom{m-\lambda}{\lambda} \frac{\Gamma(a-\lambda)}{\Gamma(a-m+\lambda)} \left(\frac{2}{x}\right)^{m-2\lambda}.$$

After expressing $\overset{m+\frac{1}{2}}{J}(x)$ by means of a trigonometric series the differential equation is obtained in an original way.

$\bar{J}^a(x)$ is shown to be a second particular integral; but $\mathring{J}(x)$, $\bar{J}^a(x)$ not being independent when a is an integer (see (2)), Schläfli's form of the second integral is introduced

$$\mathring{K}(x) = \cot a\pi \mathring{J}(x) - \frac{1}{\sin a\pi} \cdot \bar{J}^a(x)$$

as the complementary function; it is developed in a series by the method already used for $\mathring{J}(x)$, and some pretty problems in evaluating indeterminate forms are neatly solved. The complementary function is shown to satisfy the relations already found for $\mathring{J}(x)$.

The conjugate functions $\mathring{P}(x)$, $\mathring{Q}(x)$ are now defined, and Lommel's and Weber's formulas derived. The chapter closes with a discussion of the effect which would be produced on the function by rotating x through m half circles in the positive half plane. The discussion is extended to apply to $\mathring{K}(x)$, $\mathring{P}(x)$, $\mathring{Q}(x)$.

This chapter is clearly and forcibly written; the results are obtained after an easy and natural process, without the long maze of formulas frequently found in the older treatments.

The second chapter of thirty-three pages treats of the representation of Bessel's functions by means of definite integrals. For this purpose the generalized Γ-function

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int e^u u^{-a} du$$

is used as the starting point, and this value of $\Gamma(a)$ is introduced into the first definition (1) of $\mathring{J}(x)$, which reduces to

$$\mathring{J}(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^a \int e^{u-\frac{x^2}{4u}} u^{-a-1} du = \frac{1}{2\pi i}\int e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} t^{-a-1} dt.$$

Although not all of the results are new, this chapter exhibits a great deal of Graf's own work in his successful use of the "Schleifenintegral," which may be said to be almost characteristic of his researches.*

The related functions are then treated in the same way, and several results are obtained by changing the form of the

* *Cf.*, especially, his Theorie der Γ-Funktionen and Auswertung bestimmter Integrale, and the dissertations of Schenkel and Eggenberger.

path of integration. The successive steps of this process are so clear that a student familiar with the elements of complex integration would have his ideas greatly extended and systematized by reading the chapter.

Finally, the same functions are expressed as integrals of simpler functions, the path of integration being a "Doppelumlauf" around the point ix and extending to infinity parallel to the negative part of the axis of reals. The complex variable is designated by x or by u, and the real component by t. Hyperbolic functions are distinguished from trigonometric ones by the use of the German letters, thus

$$\cosh x \text{ by } \mathfrak{cos}\, x, \quad \sinh x \text{ by } \mathfrak{sin}\, x.$$

One difficulty of this paragraph is to know just what end is in view. The reader is easily and directly carried on from form to form, often without distinctly knowing why, except, as the heading of the chapter suggests, to start from the first definition, and deduce the various definite integrals which define the same function.

The third chapter discusses the problem of determining the value of $\overset{a}{J}(x)$ for large values of x. The starting point is

$$\overset{a}{P}(x) = e^{i(a-\frac{1}{2})\frac{\pi}{2}} \frac{(2x)^{-a}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+a)} \int_{-N+ix}^{ix} e^{u}(u^2+x^2)^{a-\frac{1}{2}}\, du$$

in which N is large, hence u approaches a real negative value at the lower limit. When $x = \infty$, and the integration is reduced to straight line integrals by Cauchy's theorem,

$$\overset{a}{P}(x) \doteq \frac{e^{i\left(x-\frac{\pi}{4}\right)}}{\sqrt{2\pi x}}; \text{ similarly } Q \doteq \frac{e^{-i\left(x-\frac{\pi}{4}\right)}}{\sqrt{2\pi x}},$$

Q always being derived from P by changing the sign of i. By means of the relations between P, Q, J, K, when $x \doteq \infty$,

$$\overset{a}{J}(x) = \sqrt{\frac{2}{\pi x}} \cos\left[x - (a+\tfrac{1}{2})\frac{\pi}{2}\right],$$

$$\overset{a}{K}(x) \doteq \sqrt{\frac{2}{\pi x}} \sin\left(x - (a+\tfrac{1}{2})\frac{\pi}{2}\right).$$

Corresponding simple forms are obtained for P, Q, J, K, when x replaced by $\pm ix$, $e^{\pm\frac{i\pi}{2}}x$, $e^{\mp\frac{3\pi i}{2}}x$.

The same problem is then considered from another standpoint; a semi-convergent series is obtained for $\overset{a}{P}(x)$ and it is proved that the remainder after n terms is numerically less than the $(n+1)$th term.*

A third method is applied to the case in which x is positive. Starting with the formula

$$\overset{a}{J}(x) = \frac{1}{2\pi i}\int e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} t^{-a-1}dt,$$

a transformation is made, and the new path of integration exhibited on a two leaved surface. One formidable expression for $\overset{a}{J}(x)$ is produced, consisting of a trigonometric series with Γ-functions as coefficients, a remainder term, and two double integrals. This is the only place in the work in which a double integral is employed.

An appendix is added to this chapter in which the problem of evaluating

$$\lim_{a=\infty} \overset{a}{J}(x)$$

is discussed. Three cases are considered, according as x is less than, equal to, or greater than unity. Duhamel's and Poisson's forms are derived, and a new form

$$\lim_{a=\infty} a^{\frac{1}{3}}\, \overset{a}{J}(a) = \frac{\Gamma\left(\frac{1}{3}\right)}{\pi 2^{\frac{1}{3}} 3^{\frac{1}{6}}}$$

as dividing line between the two.

The fourth chapter obtains the derivative of $\overset{a}{J}(x)$ with regard to the parameter, and has several theorems on the roots of $\overset{a}{J}(x) = 0$. The parameter is always regarded as real. Excluding the case in which a is a negative integer, $\left[\frac{x}{2}\right]^{-a} \overset{a}{J}(x) = 0$ can never have a double root; when $a > 1$, the roots are always real; when $-2 < a < -1$, two roots are imaginary; and two more roots become imaginary for every negative integer greater than a, as a dimishes towards $-\infty$. When $-1 < a$, the positive roots (α_s) of $\overset{a-1}{J}(x)$, (β_s) of $\overset{a}{J}(x)$ and (γ_s) of $\overset{a+1}{J}(x)$ satisfy the relations

* Cf. Rayleigh, Theory of Sound, for a similar remark for $\overset{a}{J}(x)$.

$$0 < a_1 < \beta_1 < \gamma_1 < a_2 < \beta_2 < \gamma_2 < \cdots < a_s < \beta_s < \gamma_s.$$

These numerous important results are clearly and precisely obtained in ten pages, with intervening steps sufficiently indicated to enable one to read the matter readily.

The last chapter discusses the development of functions with Bessel's functions for coefficients. After discussing the convergence of a particular series, the general form is assumed

$$f(x) = \sum_{\lambda=1}^{\infty} A_\lambda \cdot \overset{*}{J}(B_\lambda x) \qquad [\overset{*}{J}(B_\lambda) = 0]$$

and the A_λ are to be obtained according to the process employed in Fourier's series by means of the theorem

$$\int_0^1 \overset{*}{J}(B_\kappa x)\overset{*}{J}(B_\lambda x)dx = 0.$$

The result (Schlömilch's theorem) is

$$f(x) = \sum_{m=1}^{\infty} \int_0^1 \frac{2t\overset{*}{J}(B_m x)\overset{*}{J}(B_m t)f(t)}{[\overset{a+1}{J}(B_m)]^2} dt.$$

A discussion of several pages concerning the convergence of this series closes the volume; although the details are not hard to follow, it is by no means clear just what conditions must be fulfilled to have the series represent the function.

Copious references are given throughout, and a list of some fifty memoirs is added, though most results previously obtained were derived in quite a different way.

On the whole the book has a fresh and original appearance; Chapters I and IV especially are, in the reviewer's opinion, the most readable presentation extant of the subject treated.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
October 18, 1898.

SHORTER NOTICES.

Leçons de Cosmographie. By F. TISSERAND and H. ANDOYER. Paris, Colin & Cie. 8vo, 369 pp., with 12 plates.

THIS is an excellent text book of its kind on elementary astronomy. It is written on the plan usually adopted in such books, as, for example, Young's General Astronomy, but it contains much less detail and information. It seems to be intended for the class of student who wishes to get some knowledge of the subject but who does not wish to go much further. The explanations are easy to follow and clearly set forth, and the reading of it will be found to furnish pleasure and interest to the student. A very slight knowledge of plane and spherical geometry and of algebra and trigonometry is all that is required to follow the demonstrations.

The main part of the volume consists of six books, occupying 237 pages, respectively devoted to general spherical astronomy, the earth, sun, moon, planets and stars. Most of the important phenomena are touched upon, though necessarily in many cases very slightly; so much space being taken up with the older astronomy, the information concerning modern physical researches has to be considerably curtailed. Next follow 30 pages of history and 100 pages containing five "Notices" reprinted from the later volumes of the *Annuaire du Bureau des Longitudes*. The latter seem somewhat out of place as they contain detailed discussions of special points which would hardly be appreciated by the class of student for whom the earlier part of the book appears to be intended.

Definite statements are of course necessary in an elementary text book to give the beginner confidence, but occasionally the authors have gone rather too far in this direction: for example, on p. 64 where the density of the earth at its center is given, without a qualifying remark, as 10.6, and on p. 82 where an inaccurate definition of the seasons is hardly made clear by the remarks on p. 93. On the other hand the historical and descriptive parts are well written and we are glad to find a chapter devoted to the explanation of map construction. There are twelve illustrative plates most of which are taken from the best recent photographs.

ERNEST W. BROWN.

Annuaire pour l'An 1899, publié par le Bureau des Longitudes. Paris, Gauthier-Villars et Fils.

THIS little annual, with its mass of scientific information for a franc and a half and a slip containing a ready written notice for the reviewer, appears as usual in good time. One can hardly glance through its pages, filled as they are with numerical details of all kinds, without being impressed with the industry of scientific workers of this century. It would be an interesting employment for some statistician to make an estimate of the number of hours of work which have been devoted to obtaining the numerical results here given.

We note one or two changes from last year. The astronomical distances of the sun and moon from the earth have been altered so as to correspond with the solar parallax 8″.80; this value is probably very close to the true one. The table of small planets has been reduced in size by the omission of certain details. These and other changes necessary to bring the volume up to date will be found noted in the preface.

The four notices which accompany the volume are of more interest to the student of terrestrial and celestial phenomena than to the mathematician. The first, "Sur les ballons-sondés" by M. Bouquet de la Grye, tells mainly of the attempts which have lately been made by means of balloons and kites to obtain more information of the upper regions of the atmosphere. The second, "La géodésie moderne en France" by M. Bassot, contains an account of the rise, the decline, and the present activity of the French geodetic service. The third, "Note sur le sidérostat à lunette de 60^{m} de foyer et 1^{m}.25 d'ouverture, en construction chez M. P. Gautier," is an illustrated account of the large refracting telescope which is at present under construction and which will form one of the chief attractions at the Paris Exhibition of 1900. In the last, "Sur les travaux exécutés à l'observatoire du mont Blanc en 1898," M. J. Janssen reminds us that the snow-bound observatory is being made use of, notwithstanding the many difficulties encountered by the hardy observers who venture to stay there for a short time each summer.

ERNEST W. BROWN.

L'Ora Esatta Dappertutto, etc. By MICHELE RAJNA. Ulrico Hoepli, Milan.

THIS is a little volume of 130 pages devoted to simple explanations of the measurement of time. It contains maps and tables by means of which the true local time of

any place in Italy can be found when the hour of central Europe (used for civil purposes in that country) is known. Much astronomical information is also given. The map of Italy, which is divided into four parts for convenience, deserves especial commendation. The geographical details are very full and red lines drawn at intervals of a quarter of a degree of longitude, or one minute of time, show at a glance the local time difference between any two places.

ERNEST W. BROWN.

Formulario scholastico di matematica elementare. Per MARCO AURELIO ROSSOTTI. Milan, M. Hoepli, 1899. 18mo, pp. xvi + 191.

THIS compendium is prepared for use in the secondary schools of Italy; in the author's opinion it is representative of their methods, needs, and requirements; it thus possesses interest to the American reader in addition to the intrinsic value of its material. Its contents fall into four parts occupied with arithmetic, algebra, geometry, and trignometry, respectively. As is indicated by the title, formulæ and results alone are given; the book is in no sense didactic and all demonstrations are excluded.

The first part contains numerous tables, among which are those of all primes less than ten thousand; of the minimum divisors of integers less than ten thousand and not divisible by two, three, five, or eleven; of the eight perfect numbers less than the twenty-fourth power of ten; of the first twenty-six pairs of amicable numbers; of the squares and cubes of all integers from one to one thousand; of the first fifty powers of two, three, and five; of the square and cube roots, to five places of decimals, of all the integers up to one thousand. The great convenience to be gained by the universal adoption of the metric system in Italy is evidenced by the fact that twelve pages of the seventy devoted to this part are sacrificed to the different tables of weights and measures of twelve Italian states. This section concludes with various theorems in ratio and proportion, and the rules of interest, alligation, and false position.

The second part presents initially the laws of the operations of ordinary algebra. There is a curious inversion in notation when $b + ai$ appears as the symbol of the complex number. The elaborate tabular discussions of the solutions of linear and quadratic equations are so exhaustive as to be almost painful in their details. The usual properties of progressions, permutations, probabilities, indeterminate, exponential, and binomial equations are accompanied by col-

lections of theorems relative to inequations, maxima and minima, and continued fractions, together with an eleven-place table of decimal logarithms of the first one hundred and fifty prime numbers (the last one is eight hundred and fifty-nine). The binomial theorem of Newton terminates this part, which concludes with the one hundredth paragraph and the one hundred and thirteenth page of the book.

The first eighteen sections of the part dealing with geometry are taken up with the elements of the simple regular and irregular polygons. The details of the geometry of the circle and the computation of π follow; the series of Wallis, Leibnitz, Brounker, Lacroix, and Bernouilli, and the value of π to one hundred places of decimals are given. The remainder of the third part is devoted to the geometry of space and the sphere, concluding with a concise résumé of the characteristic properties of the conic sections.

The extent of the syllabus of trigonometry, which fills the last forty-two pages of the book, is indicated by the presence of De Moivre's theorem and its corollaries and the elements of the circles associated with a spherical triangle.

The above volume is the two hundred and eightieth number of the scientific series of the collection of manuals now in course of publication by M. Hoepli, of Milan. Up to the present there have been issued six hundred different numbers of these manuals devoted to subjects in science, technology, literature, law, and art. Of the various volumes relating to mathematics the following may be mentioned: Aschieri's analytical geometry, descriptive geometry, and projective geometry; Bagnoli's statics; Cattaneo's thermodynamics; Panizza's practical arithmetic, rational arithmetic, exercises in arithmetic; Pascal's differential calculus, integral calculus, calculus of variations and finite differences, exercises in the infinitesimal calculus, determinants and their applications, elliptic functions, repertorium of higher mathematics; Pincherle's elementary algebra, algebraic analysis, exercises in algebra, elementary geometry, exercises in geometry; Scarpis's theory of numbers; and translations of Ball's mechanics and Lockyer's astronomy.

EDGAR ODELL LOVETT.

Lectures on Elementary Mathematics. By JOSEPH LOUIS LAGRANGE. Translated by THOMAS J. MCCORMACK. Chicago, The Open Court Publishing Company, 1898. xvi + 156 pp.

THE English language is notably deficient in translations into it of standard or classical mathematical works, ancient

and modern, first published in other tongues. To mention only one out of many, Gauss's masterpiece, the Disquitiones Arithmeticæ, which has been extant for some time in excellent translations into French and German, still seeks a translator into English. Lagrange himself (p. 58 of the above translation) pointed out the need of translations of the mathematical works which the Greeks have left us. Since Lagrange's time quite a little has indeed been done to meet this want by translations into modern languages, but here also English lags far behind. It is therefore gratifying to note somewhat of a willingness springing up among American mathematicians to enrich the mathematical literature in English, by translations. Among those which have appeared or have been announced in recent years, not the least welcome is that which we shall consider a little more in detail in the following lines. We note also in passing (p. 152) that a translation (by Professors Beman and Smith) of Fink's Geschichte der Elementar-Mathematik (Tübingen, 1890) is soon to a ppear; while Fink's little book cannot claim rank as a standard work, it still sketches the development of mathematics in a clear and attractive style, and a translation may well find room in our language, and be of distinct service in stimulating that interest in the history of mathematics, and in the careful study of the development of its theories, which the straws upon the current of the mathematical stream indicate to be gaining a foothold among us.

The translation before us is made from the "Leçons élémentaires sur les mathématiques," delivered in the year 1795 at the École Normale. A well-written and interesting biographical sketch of Lagrange by the translator (7 pp.) precedes the work proper whose contents and scope may be sufficiently delineated by giving the titles of the various lectures:

I. "On arithmetic, and in particular fractions and logarithms;" II. "On the operations of arithmetic;" III. "On algebra, particularly the resolution of equations of the third and fourth degree;" IV. "On the resolution of numerical equations;" V. "On the employment of curves in the solution of problems."

It is only necessary to note in addition that both the questions taken up and their treatment are elementary even from the standpoint of a century ago, when the algebraic unsolvability of equations of degree greater than four was not yet proved, and when the powerful theory of substitution groups, towards which Lagrange himself was tending, was

still unknown. Of course, these lectures are most interesting to those to whom the subject matter is already quite familiar; still there is little, if any, demand made upon the store of mathematical knowledge of the reader which a good and thorough high school course would not supply. Nevertheless, the presentations of even elementary topics by master minds are always instructive, and in this case unusually attractive as well. Historical and methodological remarks abound, and are so woven together with the mathematical material proper, and the whole is so vivified by the clear and almost chatty style of the author as to give the lectures a charm for the reader not often to be found in mathematical works. The translation is well done, the publishers have presented it in appropriate form, and the work deserves a wide circle of readers. A short note by the translator on the origin of algebra, and a good index close the work, which bears a portrait of Lagrange as frontispiece.

It is to be hoped that translations into English of Lagrange's larger work on equations ("Traité de la résolution des équations numériques de tous les degrés," first edition, 1798, third edition, with analysis of the work by Poinsot, approved by Lagrange, 1826) and of Euler's Algebra with Lagrange's annotations, may follow.*

J. W. A. YOUNG.

L'Enseignement Mathématique is the title of a new journal of which the first number has just appeared in Paris. The editors are MM. C. A. Laisant of Paris and H. Fehr of Geneva; the publishers are Carré and Naud, 3 rue Racine, Paris. There will be six numbers a year, each containing about 80 pages octavo; the annual subscription price is 15 francs within the postal union.

While published entirely in the French language, this magazine is given an international character by a "comité de patronage" consisting of twenty mathematicians representing fifteen different nationalities. The object of the new journal is to contribute to the improvement of mathematical instruction by making more widely known its organization in different countries, by discussing methods of

* *Editorial Note.*—A translation of Euler's Elements of Algebra with notes of Bernoulli and additions of Lagrange was published by Rev. John Hewlett. The fifth edition (Longmans, Green & Co.) bears the date 1840. The preface explains that the translation was mainly due to Francis Horner. The book seems to be rare, and a new edition may be desirable.

teaching, programmes and requirements, philosophical and pedagogical questions, etc.

The contents of the first number indicate the way in which it is proposed to accomplish this object. After an introductory article by the editors, explaining the general scope of the new publication, we find articles on mathematics in Spain, by Professor Galdeano; on mathematical terminology, by M. Laisant; on scientific pedagogy, by Professor A. Binet, director of the psychological laboratory at the Sorbonne; on the teaching of "mathématiques spéciales" in France, by M. H. Laurent; on the teaching of elementary trigonometry, by Professor H. Fehr; on the teaching of the theory of vectors, by Professor Fontené. Besides these original papers there are sub-divisions devoted to notes and news, to correspondence, to bibliography and reviews. Among the latter, the review by Professor Greenhill of Professor Appell's "Éléments d'analyse mathématique(cours de l'École centrale)" may be particularly noted.

The editors call for coöperation on all persons interested in the teaching of mathematics throughout the civilized world and promise to attend, if necessary, to the translation into French of contributions written in other languages. It is to be hoped that this appeal will meet with a hearty response. Mathematical teaching is hardly anywhere organized as thoroughly and efficiently as in France; indeed, there is perhaps somewhat too much of organization and system. Just the opposite would seem to be true for most of the States of the Union; comparing notes on these questions may not be without advantage on both sides.

ALEXANDER ZIWET.

NOTES.

ATTENTION is called to a recent change in the By-Laws of the AMERICAN MATHEMATICAL SOCIETY, permitting members to commute their annual dues by a single payment of $50. Until May 1, 1899, the treasurer will accept such payment from intending life members who have already paid the dues for 1898. New members joining the Society and desiring to begin as life members, are required to make a single payment of $55 to cover the initiation and life membership fees.

THE fifth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Northwestern University, Evanston, Illinois, on Saturday, April 1, 1899. Titles and abstracts of papers to be presented should be sent to Professor T. F. Holgate, secretary of the Section, for the use of the Programme Committee not later than March 11th.

THE Association of German Scientists and Physicians will meet next September in Munich. Over two thousand members were present at the meeting at Düsseldorf in 1898.

THE London Mathematical Society held a regular meeting on the 8th of December, 1898, at which Major P. A. MACMAHON communicated a discovery which he had made in the theory of compound partitions; Mr. J. E. CAMPBELL read a paper "On simultaneous partial differential equations; Lieut. Col. A. J. C. CUNNINGHAM presented some notes on three very high primes. The following papers were communicated in abstract: "On hyper-plane coördinates," by Mr. W. H. YOUNG; "Two problems of wave propagation at the surface of an elastic solid" and "The influence of gravity on waves in an elastic solid, with special reference to the Earth," by Mr. T. J. BROMWICH: "On a theorem in determinants allied to Laplace's," by Mr. W. H. METZLER. Among the new members elected at this meeting is Professor F. MORLEY, of Haverford College.

THE three concluding volumes of the "Encyclopædia of the mathematical sciences," edited by H. BURKHARDT and W. FR. MEYER (see BULLETIN, 2d series, vol. 3, p. 326, vol. 4, p. 32, vol. 5, p. 109, pp. 151, 202) will be made up in the following manner:

Volumes IV and V:—*A*. Applications of Mathematics to Mechanics and Technical-Mechanical Problems. 1. Fundamental Principles; 2. Kinematics; 3. Kinematics of machines; 4. Statics; 5. Dynamics of a material point, including the theory of Jacobi and Hamilton; 6. Dynamics of rigid bodies and connected systems; 6*a*. Ballistics; 7. Elasticity; 8. Rigidity, plasticity, pressure; 9. Technical statics; 10. Technical dynamics; 11. Hydrodynamics; 12. Hydraulics; 13. Capillarity, cohesion.—*B*. Applications to Physics and Physical-Technical Problems. 1. Thermodynamics; 2. Conduction of heat; 3. Heat engines; 4. Kinetic theory of matter; 5. Molecular structure, crystallography, symmetry of the physical differential equations; 6. Electrostatics, magnetostatics, steady galvanic

currents; 7. Electrodynamics, electric and electromagnetic induction; 8. General theory of electromagnetism; 9. Electrotechnics; 10. Electromagnetic theory of light; 11. Optical waves; 12. Optical rays, optical instruments; 13. Emission, absorption, photometry.—*C.* Applications to Astronomy and Geodesy:—1. Spherical astronomy, (*a*) Coördinates and transformation of coördinates, (*b*) Corrections on account of precession, nutation, parallax, aberration, and refraction, (*c*) Determination of geographical position, navigation; 2. Geodesy; 3. Theoretical astronomy, determination of an orbit from observations; 4. Physical astronomy, calculation of perturbations, (*a*) Classical methods, (*b*) Recent attempts at reform. Volume VI:—*A.* History, Philosophy, Didactics.—*B.* A Review of the Development of the Mathematical Sciences in the Nineteenth Century.—*C.* Index.

RECENT advices from the Vatican state that the Abbé COZZA LUZZI, assistant librarian, has found Galileo's original manuscript treatise on the tides. The manuscript is in Galileo's handwriting and concludes with the words:—"Written in Rome in the Medici Gardens on January 8, 1616." The currently accepted text, the original of which was supposed to have been lost, differs considerably from that of the manuscript just found. Pope Leo XIII has taken the greatest interest in the discovery and has ordered the manuscript to be published in an elegant edition at the expense of the Vatican.

AMONG the papers presented at the last annual meeting of the Astronomische Gesellschaft, held at Budapest, is one by Professor R. VON KÖVESLIGETHY, of the University of Budapest, entitled "Ueber die beiden Parametergleichungen der Spectral Analyse," the object of which is to develop a mathematical theory for astrophysics. The author shows how the two fundamental equations of heat are destined to play as important a rôle in astrophysics as the principles of mechanics in astronomy. To this end the equation of emission is formed very simply in terms of the wave length and of two parameters which depend on the nature of the bodies. The author finds that his theory represents perfectly a series of bolometric measurements of the spectrum; he further insists that his analysis is applicable to important questions in astronomy, such as the determination of parallaxes, and the volume and density of the heavenly bodies.

THE Prussian Academy of Sciences has undertaken the publication of the mathematical correspondence of Leibnitz

under the editorship of Professor C. J. GERHARDT. The correspondence is to be reproduced in three royal octavo volumes, of which the first, consisting of seven hundred and sixty-eight pages, has just appeared from the press of Mayer and Müller, of Berlin. The first volume contains Leibnitz's correspondence with Oldenburg, Newton, Collins, Conti, Tschirnhaus, and Huygens; the editor of the work regards the material of this volume as sufficient to establish the claims of Leibnitz to the priority in the discovery of the infinitesimal calculus. Following the publication of his method and its use by the continental mathematicians, Leibnitz's correspondence in reply to requests for explanations and further exposition was voluminous and extensive; the correspondence of this period, which contains many unpublished discoveries of Leibnitz, will be incorporated in the second and third volumes, which may be expected to be rich in facts relative to the history of mathematics in the second half of the seventeenth and the first half of the eighteenth century.

A MANUSCRIPT treatise on geometry by Lobatschewsky of earlier date than any of his published works has recently been found at Kasan. Its publication will be undertaken by the Physico-Mathematical Society of Kasan.

THE press of Harper and Brothers announces the publication of the first number of Harper's Scientific Memoirs, a series of handbooks in science, under the general editorship of Dr. JOSEPH S. AMES, professor of physics in Johns Hopkins University. The series will embrace a large number of translations and reprints of the most important articles which have been written in the history of science, including many which have an important bearing upon the present state of science. It is proposed to print these papers with accompanying figures in thin octavo volumes, which shall be convenient for reference and yet inexpensive. The volumes will be printed in the English language, special attention being given to the translation or reproduction. Notes, explanations, and corrections will be added where needed, and references will be given from time to time to contemporary or later work. A bibliography and index will close each volume. The associate editors of the series for subjects in mathematics, mechanics, and thermodynamics are Professors W. F. MAGIE, of Princeton University, C. BARUS, of Brown University, T. C. MENDENHALL, of Worcester Polytechnic Institute, and A. G. WEBSTER, of Clark University.

AN ENGLISH translation of the second Italian edition of Loria's "Il passato ed il presente delle principali teorie geometriche," with a preface by the author, will soon be published by Professor GEORGE BRUCE HALSTED, of the University of Texas.

THE mathematical publications of the firm of Leach, Shewell, and Sanborn, have been transferred to the publishing house of D. C. Heath and Company.

PROFESSOR E. MOUGIN, of the lycée de Laval, has issued a second edition of his "Nouvelles tables de logarithmes à cinq décimales," with the imprint of the press of the author and the date 1899. These tables contain: 1° the logarithms of the numbers from 1 to 10,000, with differences and proportional parts, in nine pages (1,111 logarithms on a page whose size is $6\frac{1}{2}$ by $7\frac{1}{2}$ inches), so arranged that the initial significant figure of the number indicates the page on which the logarithm is to be found, *e. g.*, 6,725, page 6; 2° in eighteen pages the logarithmic sines, cosines, tangents, and cotangents for every minute of the first quadrant; 3° explanatory notes accompanied by numerous illustrative examples; 4° the elementary formulæ of mathematics and physics on a sheet attached by a thread so that the sheet may be removed at the time of examination and replaced later. The economy of space exhibited by the ingenious construction of these tables is further shown by the fact that while the classic tables employ more than ninety thousand figures to express the logarithms of numbers to five places of decimals, Mougin uses fewer than thirty thousand.

AT the recent anniversary meeting of the Royal Society of London the Copley Medal was awarded to Sir WILLIAM HUGGINS for his great achievements in the application of spectrum analysis to the heavenly bodies; the Rumford Medal was given to Professor OLIVER J. LODGE in recognition of his researches on radiation and on the relations between ether and matter; one of the Royal Medals was bestowed upon the Reverend JOHN KERR as the author of extremely important experimental researches in the optical relations of electricity and magnetism; and the Darwin Medal was received by Professor KARL PEARSON in recognition of the great biological importance of his work upon the theory of probability and its relation to vital statistics.

THE Paris Academy of Sciences, at its annual meeting held on December 19, 1898, awarded its prizes for the year 1898, among which are the following: the Grand Prize of

the mathematical sciences to M. ÉMILE BOREL for his memoir on the rôle of divergent series in analysis, M. MAURICE SERVANT being awarded an honorable mention. The Bordin Prize was not awarded; the Francœur Prize was given to M. VASCHY, and the Poncelet Prize to M. J. HADAMARD. The Extraordinary Prize in mechanics of six thousand francs was divided among MM. BAULE, G. CHARPY, THIÉBAUT, L. RAVIER, and MOISSENET; the Montyon Prize was awarded to M. DE MAS, a Fourneyron Prize to M. C. BOURLET, another being divided between MM. E. CARVALLO and JACOB, and a very honorable mention to M. SHARP. The Lalande Prize in astronomy was given to Dr. S. C. CHANDLER, for his researches on the variation of latitude and on variable stars, M. CHOFARDET receiving an encouragement; the Damoiseau Prize was awarded to Dr. G. W. HILL for his numerous memoirs in mathematics and astronomy; the Valz Prize was given to M. P. COLIN, and the Janssen Prize to M. BELOPOLSKY.

The prizes offered by the Academy for the next two years include the following in mathematical subjects: To be awarded at the annual meeting in December, 1899:—The Bordin Prize for the most successful memoir on the questions relative to the determination, with regard to properties and applications, of systems of orthogonal curvilinear coördinates in n variables, including as precise as possible an indication of the generality of these systems; the Francœur Prize for works and discoveries useful to the progress of pure and applied mathematics; the Poncelet Prize for the most useful work to the progress of pure or applied mathematics; the Extraordinary Prize of six thousand francs for progress of a nature to increase the efficiency of the French naval forces; the Montyon Prize in mechanics; the Plumey Prize for the best contribution to the progress of steam navigation; the Lalande and Damoiseau Prizes in astronomy, and the Montyon Prize in statistics. To be awarded at the annual meeting in December, 1900:—the Grand Prize of the mathematical sciences for the memoir perfecting, in some important point, the investigation of the number of classes of quadratic forms, having integral coefficients, in two indeterminates; the Bordin Prize for developing and perfecting the theory of surfaces applicable on the paraboloid of revolution; the Damoiseau Prize for working out the theory of one of the periodic comets of which several returns have been observed. All works and manuscripts submitted in competition for these prizes should reach the secretary of the Institute of France before the first of Oc-

tober of the year in which the prize is to be awarded. The customary regulations governing such prize competitions obtain, especially the one requiring the author of a manuscript to conceal his identity by adopting a device, motto, or nom-de-plume as a means of future identification.

We learn from *Nature* that the gold medal for the solution of the prize problem set by the faculty of mathematics and natural science of the University of Heidelberg has been awarded to Mr. Ernest Stevens, of Brighton, England, for his determination of the velocities of various gases and vapors at different temperatures.

Oxford University. The preliminary announcements of lectures for the Hilary term 1899, include the following in mathematical subjects:—By Professor W. Esson: Synthetic geometry of conics, two lectures per week; Synthetic geometry of cubics, one lecture per week.—By Professor H. H. Turner: Elementary mathematical astronomy (continued), two lectures per week.—By Professor E. B. Elliott: Elements of elliptic functions, two lectures per week.—By Mr. J. Walker: Optics, two lectures per week.—By Mr. J. C. Alsop: Selected portions of mechanics and physics, three lectures per week.—Mr. E. S. Craig will deliver two lectures per week for the Reverend F. J. Jervis-Smith, the university lecturer in mechanics.

Professor Felix Klein, of Göttingen, has been made a member of the Bavarian Maximilian Order of Science and Art.

Dr. R. W. Willson has been appointed assistant professor of astronomy at Harvard University. Dr. V. Varicak has been promoted to an assistant professorship of mechanics at the University of Agram.

At the University of Leipzig, Professor W. Scheibner, the senior professor of mathematics, recently celebrated his fifty-year doctor jubilee. Professor G. Wiedemann, for many years at the head of the department of experimental physics, will retire at Easter of this year.

It is proposed to erect a monument in memory of Félix Tisserand, Member of the Institute of France, at Nuits Saint-Georges (Côte d'Or), his native place. Subscriptions will be received by M. Desmazures, Receveur Municipal, Nuits Saint-Georges; M. Fraissinet, Observatory of Paris; M. Ragot, rue Colson, Dijon.

PROFESSOR E. O. KENDALL, of the University of Pennsylvania, died recently at the age of eighty-two years. Professor KENDALL was elected to the chair of mathematics and astronomy in the University of Pennsylvania in 1855; in 1892 he was appointed Flower professor of astronomy; in 1895 he resigned the Flower professorship in favor of Professor C. L. Doolittle, but retained the Scott professorship up to the time of his death; he was made Vice-Provost of the University in 1893, and since 1894 he was Honorary Dean of the College and Honorary Vice-Provost.

PROFESSOR BARTHOLOMEW PRICE, master of Pembroke College, Oxford University, died December 29, 1898, at the age of eighty-one years. Professor PRICE resigned the Sedleian professorship of natural philosophy in Oxford University last year after forty-five years' tenure of the chair. He is perhaps best known by his treatise on the infinitesimal calculus.

THE deaths are announced of Mr. EDWIN DUNKIN, F.R.S., the distinguished astronomer, and of Dr. W. JAJACKOWSKI, professor of mathematics in the technical high school at Lemberg.

THE seventieth anniversary, August 23, 1899, of the birth of Professor MORITZ CANTOR, of the University of Heidelberg, will be celebrated by the publication of a Festschrift, under the editorship of Professor MAX CURTZE, of Thorn. The collaborators in the preparation of this anniversary volume, the first pages of which are already in the press of Teubner, are Messrs. Braunmühl, Cajori, Curtze, Dickstein, Eneström, Favaro, Gelcich, Gow, Graf, Günther, Heath, Heiberg, Hultsch, Hunrath, Lindemann, Loria, F. Müller, Mansion, Nagl, Schubert, Staigmüller, Steinschneider, A. Sturm, Suter, P. Tannery, Unger, Vailati, Wertheim, Wohlwill.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BJERKNES (V.). Zur Theorie gewisser Vectorgrössen. Christiania, 1898. 8vo. 27 pp. M. 1.50

MCAULAY (A.). Octonions. Development of Clifford's biquaternions. Cambridge University Press, 1898. 8vo. 10s 6d.

MASCHKE (H.). Bestimmung aller ternären und quaternären Collineationsgruppen, welche mit symmetrischen und alternirenden Buchstabenvertauschungsgruppen holoedrisch isomorph sind. (*Mathematische Annalen*, 1898.) Leipzig, 1898. 8vo. 46 pp.

MAUPIN (G.). Opinions et curiosités touchant la mathématique, d'après les ouvrages français des XVI^e^, XVII^e^, et XVIII^e^ siècles. Paris, Carré et Naud, 1898. 8vo. 207 pp.

PUCHBERGER (E.). Eine allgemeinere Integration der Differentialgleichungen. Heft VI. Vienna, C. Gerold's Sohn, 1898. 8vo. 51 pp. M. 1.60

REYE (T.). Die Geometrie der Lage. Vierte, verbesserte und vermehrte Auflage. (In 3 Abteilungen.) Abteilung I. Leipzig, 1898. 8vo. M. 8.00

VINOT (J.). Récréations mathématiques. Nouveau recueil de questions curieuses et utiles extraites des auteurs anciens et modernes. Quatrième édition. Paris, Larousse, 1898. 8vo. 8 and 215 pp. Fr. 3.00

WERNICKE (A.). Die mathematisch-naturwissenschaftliche Forschung in ihrer Stellung zum modernen Humanismus. Vortrag. Berlin, Salle, 1898. 4to. 18 pp. M. 1.00

II. ELEMENTARY MATHEMATICS.

BERGMANS (C.). Compléments d'arithmétique et d'algèbre. Deuxième édition. Gand, 1898. 8vo. 294 pp. Fr. 5.50

BROUET (V.) et HAUDRICOURT (F.) et (A.). Leçons et devoirs d'arithmétique et de système métrique, à l'usage des élèves des cours supérieurs et complémentaires des écoles primaires. Livre de l'élève. Première édition. Paris, May, 1898. 16mo. 416 pp.

BUCK (R. C.). Manual of Algebra. London, 1898. 8vo. 166 pp. Cloth. 3s. 6d.

CANTOR (M.). Politische Arithmetik oder die Arithmetik des täglichen Lebens. Leipzig, Teubner, 1898. 8vo. 10 and 136 pp. Cloth. M. 1.80.

COMBEROUSSE (C. DE). Cours de mathématiques. Tome I. Partie 2: Algèbre élémentaire. Quatrième édition. Paris, 1898. 8vo. Fr. 6.00

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